*C H A P T E R 6*

# FUNDAMENTALS OF RESERVOIR FLUID FLOW

Flow in porous media is a very complex phenomenon and as such cannot be described as explicitly as flow through pipes or conduits. It is rather easy to measure the length and diameter of a pipe and compute its flow capacity as a function of pressure; in porous media, however, flow is different in that there are no clear-cut flow paths that lend themselves to measurement.

The analysis of fluid flow in porous media has evolved throughout the years along two fronts—the experimental and the analytical. Physicists, engineers, hydrologists, and the like have examined experimentally the behavior of various fluids as they flow through porous media ranging from sand packs to fused Pyrex glass. On the basis of their analyses, they have attempted to formulate laws and correlations that can then be utilized to make analytical predictions for similar systems.

The main objective of this chapter is to present the mathematical relationships that are designed to describe the flow behavior of the reservoir fluids. The mathematical forms of these relationships will vary depending upon the characteristics of the reservoir. The primary reservoir characteristics that must be considered include:

- Types of fluids in the reservoir
- Flow regimes
- Reservoir geometry
- Number of flowing fluids in the reservoir


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## TYPES OF FLUIDS

The isothermal compressibility coefficient is essentially the controlling factor in identifying the type of the reservoir fluid. In general, reservoir fluids are classified into three groups:

- Incompressible fluids
- Slightly compressible fluids
- Compressible fluids

As described in Chapter 2, the isothermal compressibility coefficient c is described mathematically by the following two equivalent expressions:

- In terms of fluid volume:

$$c = \frac{-1}{V}\frac{\partial V}{\partial p} \tag{6-1}$$

- In terms of fluid density:

$$c = \frac{1}{\rho}\frac{\partial \rho}{\partial p} \tag{6-2}$$

where V and ρ are the volume and density of the fluid, respectively.

### Incompressible Fluids

An incompressible fluid is defined as the fluid whose volume (or density) does not change with pressure, i.e.:

$$\frac{\partial V}{\partial p} = 0$$

$$\frac{\partial \rho}{\partial p} = 0$$

Incompressible fluids do not exist; this behavior, however, may be assumed in some cases to simplify the derivation and the final form of many flow equations.

## Slightly Compressible Fluids

These "slightly" compressible fluids exhibit small changes in volume, or density, with changes in pressure. Knowing the volume $V_{ref}$ of a slightly compressible liquid at a reference (initial) pressure $p_{ref}$, the changes in the volumetric behavior of this fluid as a function of pressure p can be mathematically described by integrating Equation 6-1 to give:

$$-c\int_{p_{ref}}^{p} dp = \int_{V_{ref}}^{V} \frac{dV}{V}$$

$$e^{c(p_{ref}-p)} = \frac{V}{V_{ref}}$$

$$V = V_{ref}\, e^{c(p_{ref}-p)} \tag{6-3}$$

where p = pressure, psia
V = volume at pressure p, $ft^3$
$p_{ref}$ = initial (reference) pressure, psia
$V_{ref}$ = fluid volume at initial (reference) pressure, psia

The $e^x$ may be represented by a series expansion as:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^2}{3!} + \cdots + \frac{x^n}{n!} \tag{6-4}$$

Because the exponent x [which represents the term c $(p_{ref}-p)$] is very small, the $e^x$ term can be approximated by truncating Equation 6-4 to:

$$e^x = 1 + x \tag{6-5}$$

Combining Equation 6-5 with Equation 6-3 gives:

$$V = V_{ref}\,[1 + c\,(p_{ref} - p)] \tag{6-6}$$

A similar derivation is applied to Equation 6-2 to give:

$$\rho = \rho_{ref}\,[1 - c\,(p_{ref} - p)] \tag{6-7}$$

where $V$ = volume at pressure p
$\rho$ = density at pressure p
$V_{ref}$ = volume at initial (reference) pressure $p_{ref}$
$\rho_{ref}$ = density at initial (reference) pressure $p_{ref}$

It should be pointed out that crude oil and water systems fit into this category.

## Compressible Fluids

These are fluids that experience large changes in volume as a function of pressure. All gases are considered compressible fluids. The truncation of the series expansion, as given by Equation 6-5, is not valid in this category and the complete expansion as given by Equation 6-4 is used. As shown previously in Chapter 2 in Equation 2-45, the isothermal compressibility of any compressible fluid is described by the following expression:

$$c_g = \frac{1}{p} - \frac{1}{z}\left(\frac{\partial z}{\partial p}\right)_T \tag{6-8}$$

Figures 6-1 and 6-2 show schematic illustrations of the volume and density changes as a function of pressure for the three types of fluids.

# FLOW REGIMES

There are basically three types of flow regimes that must be recognized in order to describe the fluid flow behavior and reservoir pressure distribution as a function of time. There are three flow regimes:

- Steady-state flow
- Unsteady-state flow
- Pseudosteady-state flow

## Steady-State Flow

The flow regime is identified as a steady-state flow if the pressure at every location in the reservoir remains constant, i.e., does not change with time. Mathematically, this condition is expressed as:

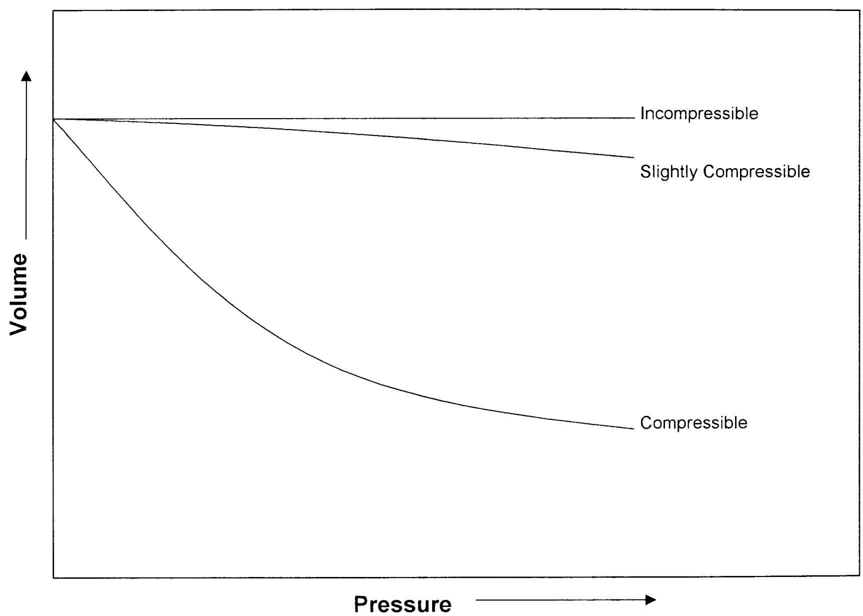



**Figure 6-1.** Pressure-volume relationship.

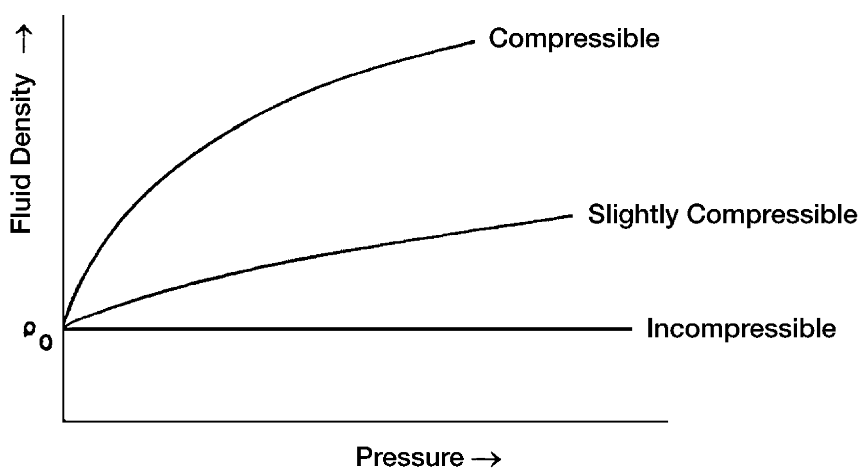



**Figure 6-2.** Fluid density versus pressure for different fluid types.

$$\left(\frac{\partial p}{\partial t}\right)_i = 0 \tag{6-9}$$

The above equation states that the rate of change of pressure p with respect to time t at any location i is zero. In reservoirs, the steady-state flow condition can only occur when the reservoir is completely recharged and supported by strong aquifer or pressure maintenance operations.

### Unsteady-State Flow

The unsteady-state flow (frequently called *transient flow*) is defined as the fluid flowing condition at which the rate of change of pressure with respect to time at any position in the reservoir is not zero or constant. This definition suggests that the pressure derivative with respect to time is essentially a function of both position i and time t, thus

$$\left(\frac{\partial p}{\partial t}\right) = f(i, t) \tag{6-10}$$

### Pseudosteady-State Flow

When the pressure at different locations in the reservoir is declining linearly as a function of time, i.e., at a constant declining rate, the flowing condition is characterized as the pseudosteady-state flow. Mathematically, this definition states that the rate of change of pressure with respect to time at every position is constant, or

$$\left(\frac{\partial p}{\partial t}\right)_i = \text{constant} \tag{6-11}$$

It should be pointed out that the pseudosteady-state flow is commonly referred to as semisteady-state flow and quasisteady-state flow.

Figure 6-3 shows a schematic comparison of the pressure declines as a function of time of the three flow regimes.

## RESERVOIR GEOMETRY

The shape of a reservoir has a significant effect on its flow behavior. Most reservoirs have irregular boundaries and a rigorous mathematical

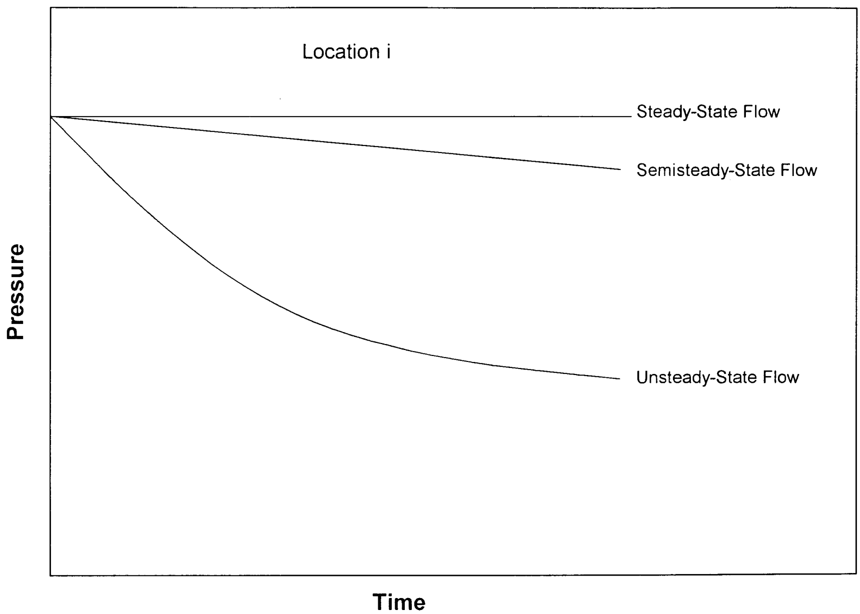



**Figure 6-3.** Flow regimes.

description of geometry is often possible only with the use of numerical simulators. For many engineering purposes, however, the actual flow geometry may be represented by one of the following flow geometries:

- Radial flow
- Linear flow
- Spherical and hemispherical flow

## Radial Flow

In the absence of severe reservoir heterogeneities, flow into or away from a wellbore will follow radial flow lines from a substantial distance from the wellbore. Because fluids move toward the well from all directions and coverage at the wellbore, the term *radial flow* is given to characterize the flow of fluid into the wellbore. Figure 6-4 shows idealized flow lines and iso-potential lines for a radial flow system.

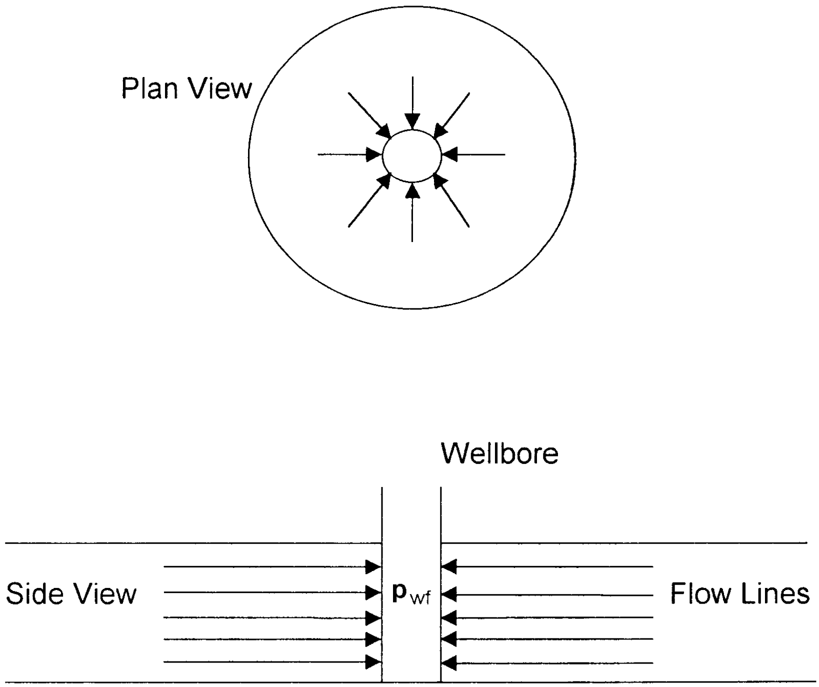



**Figure 6-4.** Ideal radial flow into a wellbore.

## Linear Flow

Linear flow occurs when flow paths are parallel and the fluid flows in a single direction. In addition, the cross-sectional area to flow must be constant. Figure 6-5 shows an idealized linear flow system. A common application of linear flow equations is the fluid flow into vertical hydraulic fractures as illustrated in Figure 6-6.

## Spherical and Hemispherical Flow

Depending upon the type of wellbore completion configuration, it is possible to have a spherical or hemispherical flow near the wellbore. A well with a limited perforated interval could result in spherical flow in the vicinity of the perforations as illustrated in Figure 6-7. A well that only partially penetrates the pay zone, as shown in Figure 6-8, could result in hemispherical flow. The condition could arise where coning of bottom water is important.

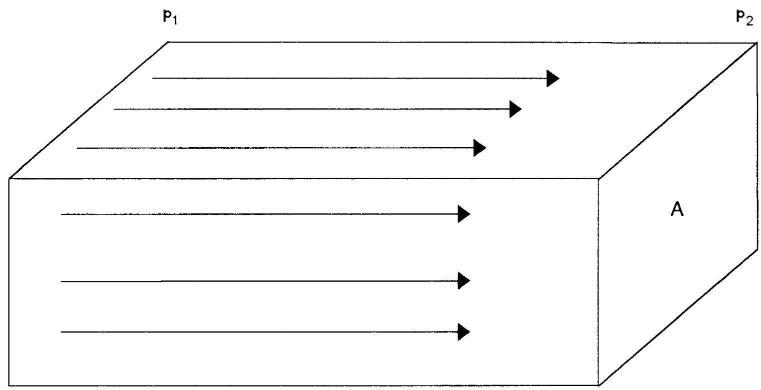



**Figure 6-5.** Linear flow.

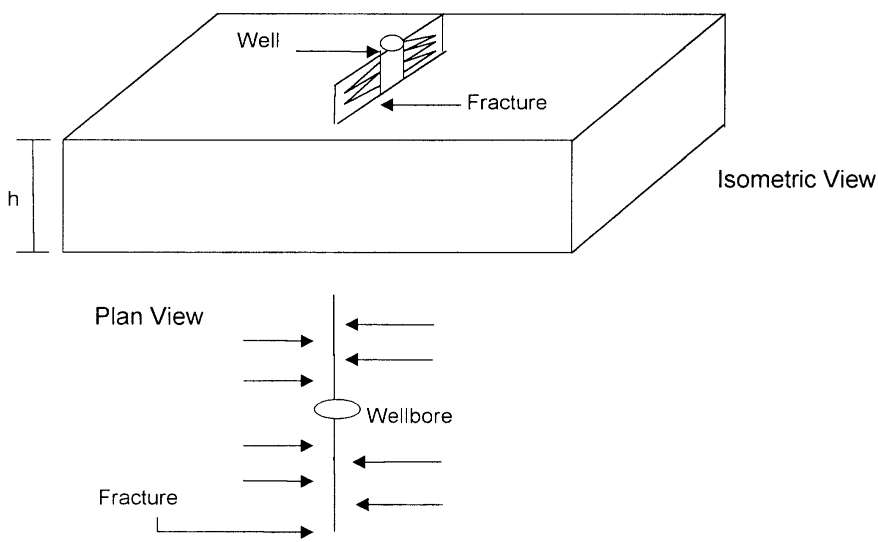



**Figure 6-6.** Ideal linear flow into vertical fracture.

## NUMBER OF FLOWING FLUIDS IN THE RESERVOIR

The mathematical expressions that are used to predict the volumetric performance and pressure behavior of the reservoir vary in forms and complexity depending upon the number of mobile fluids in the reservoir. There are generally three cases of flowing systems:

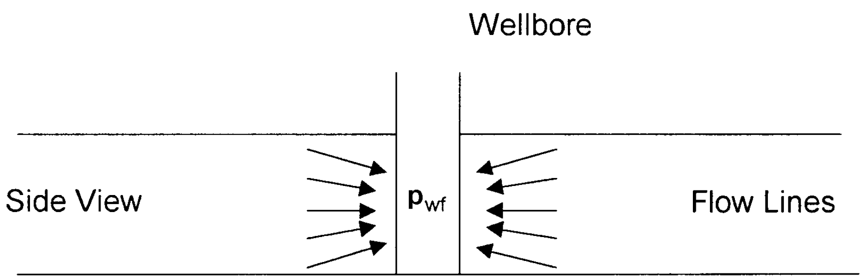



**Figure 6-7.** Spherical flow due to limited entry.

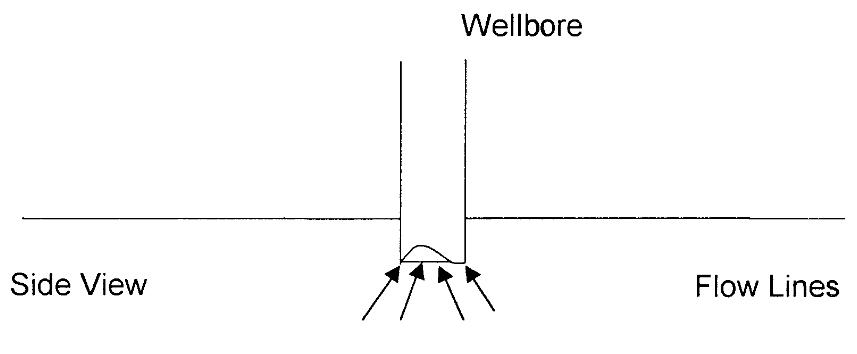



**Figure 6-8.** Hemispherical flow in a partially penetrating well.

- Single-phase flow (oil, water, or gas)
- Two-phase flow (oil-water, oil-gas, or gas-water)
- Three-phase flow (oil, water, and gas)

The description of fluid flow and subsequent analysis of pressure data becomes more difficult as the number of mobile fluids increases.

## FLUID FLOW EQUATIONS

The fluid flow equations that are used to describe the flow behavior in a reservoir can take many forms depending upon the combination of variables presented previously (i.e., types of flow, types of fluids, etc.). By combining the conservation of mass equation with the transport equation (Darcy's equation) and various equations-of-state, the necessary flow equations can be developed. Since all flow equations to be considered depend on Darcy's Law, it is important to consider this transport relationship first.

## Darcy's Law

The fundamental law of fluid motion in porous media is Darcy's Law. The mathematical expression developed by Henry Darcy in 1856 states the velocity of a homogeneous fluid in a porous medium is proportional to the pressure gradient and inversely proportional to the fluid viscosity. For a horizontal linear system, this relationship is:

$$\nu = \frac{q}{A} = -\frac{k}{\mu}\frac{dp}{dx} \tag{6-12}$$

$\nu$ is the **apparent** velocity in centimeters per second and is equal to q/A, where q is the volumetric flow rate in cubic centimeters per second and A is total cross-sectional area of the rock in square centimeters. In other words, A includes the area of the rock material as well as the area of the pore channels. The fluid viscosity, $\mu$, is expressed in centipoise units, and the pressure gradient, dp/dx, is in atmospheres per centimeter, taken in the same direction as $\nu$ and q. The proportionality constant, k, is the *permeability* of the rock expressed in Darcy units.

The negative sign in Equation 6-12 is added because the pressure gradient is negative in the direction of flow as shown in Figure 6-9.

For a horizontal-radial system, the pressure gradient is positive (see Figure 6-10) and Darcy's equation can be expressed in the following generalized radial form:

$$\nu = \frac{q_r}{A_r} = \frac{k}{\mu}\left(\frac{\partial p}{\partial r}\right)_r \tag{6-13}$$

where $q_r$ = volumetric flow rate at radius r
$A_r$ = cross-sectional area to flow at radius r
$(\partial p/\partial r)_r$ = pressure gradient at radius r
$\nu$ = apparent velocity at radius r

The cross-sectional area at radius r is essentially the surface area of a cylinder. For a fully penetrated well with a net thickness of h, the cross-sectional area $A_r$ is given by:

$$A_r = 2\pi rh$$

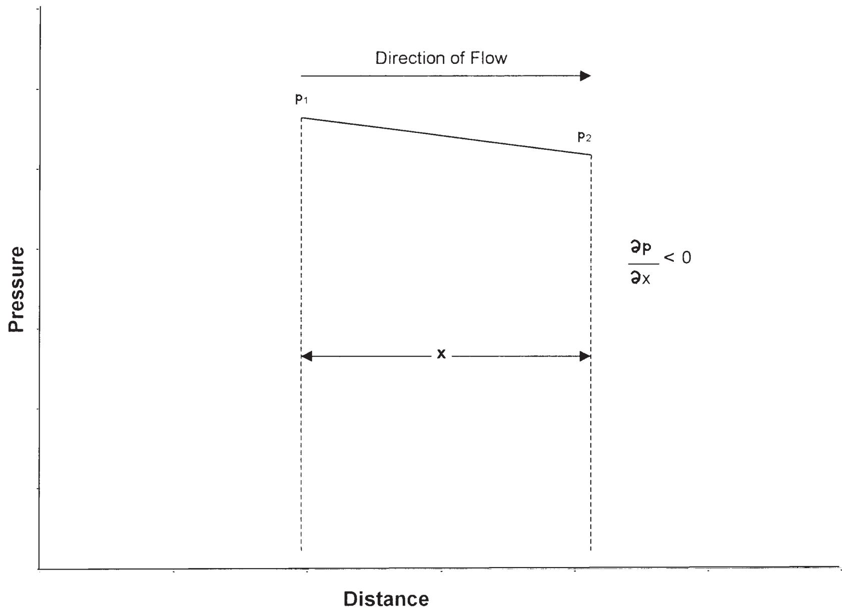



**Figure 6-9.** Pressure vs. distance in a linear flow.

Darcy's Law applies only when the following conditions exist:

- Laminar (viscous) flow
- Steady-state flow
- Incompressible fluids
- Homogeneous formation

For turbulent flow, which occurs at higher velocities, the pressure gradient increases at a greater rate than does the flow rate and a special modification of Darcy's equation is needed. When turbulent flow exists, the application of Darcy's equation can result in serious errors. Modifications for turbulent flow will be discussed later in this chapter.

## STEADY-STATE FLOW

As defined previously, steady-state flow represents the condition that exists when the pressure throughout the reservoir does not change with time. The applications of the steady-state flow to describe the flow behavior of several types of fluid in different reservoir geometries are presented below. These include:

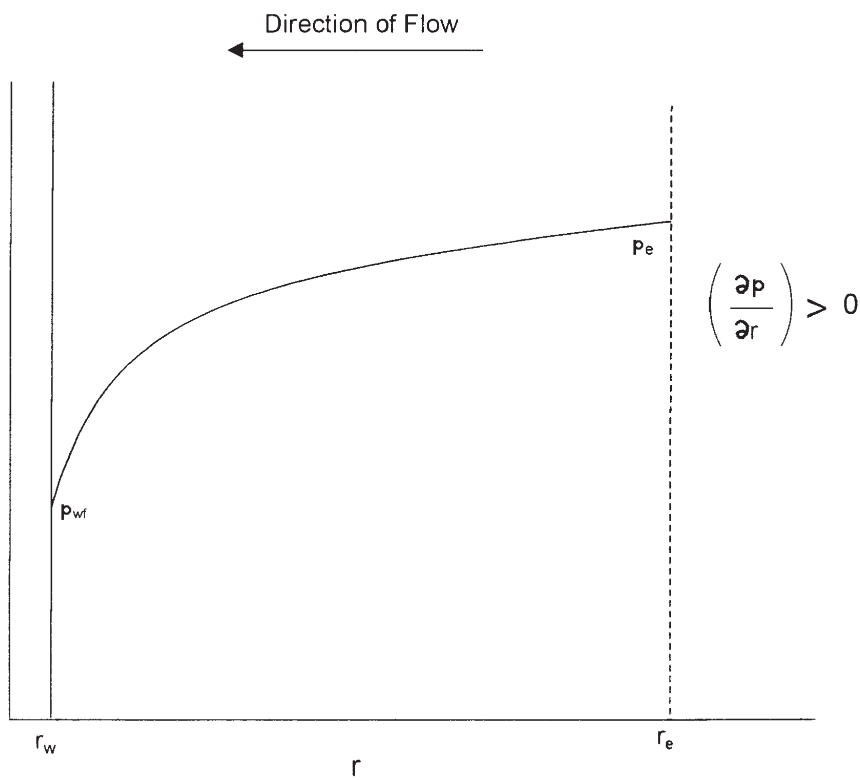



**Figure 6-10.** Pressure gradient in radial flow.

- Linear flow of incompressible fluids
- Linear flow of slightly compressible fluids
- Linear flow of compressible fluids
- Radial flow of incompressible fluids
- Radial flow of slightly compressible fluids
- Radial flow of compressible fluids
- Multiphase flow

## Linear Flow of Incompressible Fluids

In the linear system, it is assumed the flow occurs through a constant cross-sectional area A, where both ends are entirely open to flow. It is also assumed that no flow crosses the sides, top, or bottom as shown in Figure 6-11.

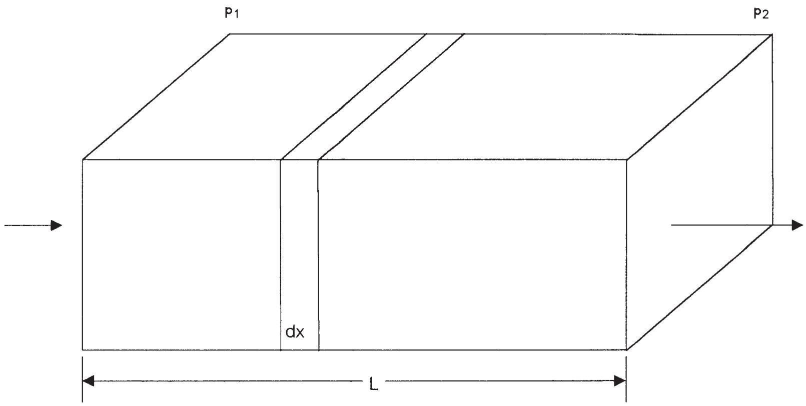



**Figure 6-11.** Linear flow model.

If an incompressible fluid is flowing across the element dx, then the fluid velocity v and the flow rate q are constants at all points. The flow behavior in this system can be expressed by the differential form of Darcy's equation, i.e., Equation 6-12. Separating the variables of Equation 6-12 and integrating over the length of the linear system gives:

$$\frac{q}{A}\int_0^L dx = -\frac{k}{\mu}\int_{p_1}^{p_2} dp$$

or:

$$q = \frac{kA(p_1 - p_2)}{\mu L}$$

It is desirable to express the above relationship in customary field units, or:

$$q = \frac{0.001127\ kA(p_1 - p_2)}{\mu L} \tag{6-14}$$

where q = flow rate, bbl/day
k = absolute permeability, md
p = pressure, psia
$\mu$ = viscosity, cp
L = distance, ft
A = cross-sectional area, $ft^2$

### Example 6-1

An incompressible fluid flows in a linear porous media with the following properties:

| | | |
|---|---|---|
| $L = 2000$ ft | $h = 20'$ | width $= 300'$ |
| $k = 100$ md | $\phi = 15\%$ | $\mu = 2$ cp |
| $p_1 = 2000$ psi | $p_2 = 1990$ psi | |

Calculate:

a. Flow rate in bbl/day
b. Apparent fluid velocity in ft/day
c. Actual fluid velocity in ft/day

### Solution

Calculate the cross-sectional area A:

$A = (h)\ (\text{width}) = (20)\ (300) = 6000\ \text{ft}^2$

a. Calculate the flow rate from Equation 6-14:

$$q = \frac{(0.001127)\ (100)\ (6000)\ (2000 - 1990)}{(2)\ (2000)} = 1.6905\ \text{bbl/day}$$

b. Calculate the apparent velocity:

$$v = \frac{q}{A} = \frac{(1.6905)(5.615)}{6000} = 0.0016\ \text{ft/day}$$

c. Calculate the actual fluid velocity:

$$v = \frac{q}{\phi A} = \frac{(1.6905)(5.615)}{(0.15)(6000)} = 0.0105\ \text{ft/day}$$

The difference in the pressure ($p_1 - p_2$) in Equation 6-14 is not the only driving force in a tilted reservoir. The gravitational force is the other important driving force that must be accounted for to determine the direction and rate of flow. The fluid gradient force (gravitational force) is always directed *vertically downward* while the force that results from an applied pressure drop may be in any direction. The force causing flow would then be the *vector sum* of these two. In practice, we obtain this result by introducing a new parameter, called fluid potential, which has the same dimensions as pressure, e.g., psi. Its symbol is Φ. The fluid potential at any point in the reservoir is defined as the pressure at that point less the pressure that would be exerted by a fluid head extending to an arbitrarily assigned datum level. Letting $\Delta z_i$ be the vertical distance from a point i in the reservoir to this datum level

$$\Phi_i = p_i - \left(\frac{\rho}{144}\right)\Delta z_i \tag{6-15}$$

where ρ is the density in $lb/ft^3$.

Expressing the fluid density in gm/cc in Equation 6-15 gives:

$$\Phi_i = p_i - 0.433\,\gamma\,\Delta z_i \tag{6-16}$$

where $\Phi_i$ = fluid potential at point i, psi
$p_i$ = pressure at point i, psi
$\Delta z_i$ = vertical distance from point i to the selected datum level
ρ = fluid density, $lb/ft^3$
γ = fluid density, $gm/cm^3$

The datum is usually selected at the gas-oil contact, oil-water contact, or at the highest point in formation. In using Equations 6-15 or 6-16 to calculate the fluid potential $\Phi_i$ at location i, **the vertical distance $\Delta z_i$ is assigned as a positive value when the point i is below the datum level and as a negative when it is above the datum level,** i.e.:

If point i is above the datum level:

$$\Phi_i = p_i + \left(\frac{\rho}{144}\right)\Delta z_i$$

and

$$\Phi_i = p_i - 0.433\,\gamma\,\Delta z_i$$

If point i is below the datum level:

$$\Phi_i = p_i - \left(\frac{\rho}{144}\right)\Delta z_i$$

and

$$\Phi_i = p_i - 0.433\,\gamma\,\Delta z_i$$

Applying the above-generalized concept to Darcy's equation (Equation 6-14) gives:

$$q = \frac{0.001127\,kA\,(\Phi_1 - \Phi_2)}{\mu L} \tag{6-17}$$

It should be pointed out that the fluid potential drop ($\Phi_1 - \Phi_2$) is equal to the pressure drop ($p_1 - p_2$) only when the flow system is horizontal.

## Example 6-2

Assume that the porous media with the properties as given in the previous example is tilted with a dip angle of 5° as shown in Figure 6-12. The incompressible fluid has a density of 42 lb/ft$^3$. Resolve Example 6-1 using this additional information.

## Solution

*Step 1.* For the purpose of illustrating the concept of fluid potential, select the datum level at half the vertical distance between the two points, i.e., at 87.15 feet, as shown in Figure 6-12.

*Step 2.* Calculate the fluid potential at Points 1 and 2.

**Since Point 1 is below the datum level, then:**

$$\Phi_1 = p_1 - \left(\frac{\rho}{144}\right)\Delta z_1 = 2000 - \left(\frac{42}{144}\right)(87.15) = 1974.58 \text{ psi}$$

**Since Point 2 is above the datum level, then:**

$$\Phi_2 = p_2 + \left(\frac{\rho}{144}\right)\Delta z_2 = 1990 + \left(\frac{42}{144}\right)(87.15) = 2015.42 \text{ psi}$$

Because $\Phi_2 > \Phi_1$, the fluid flows downward from Point 2 to Point 1. The difference in the fluid potential is:

$$\Delta\Phi = 2015.42 - 1974.58 = 40.84 \text{ psi}$$

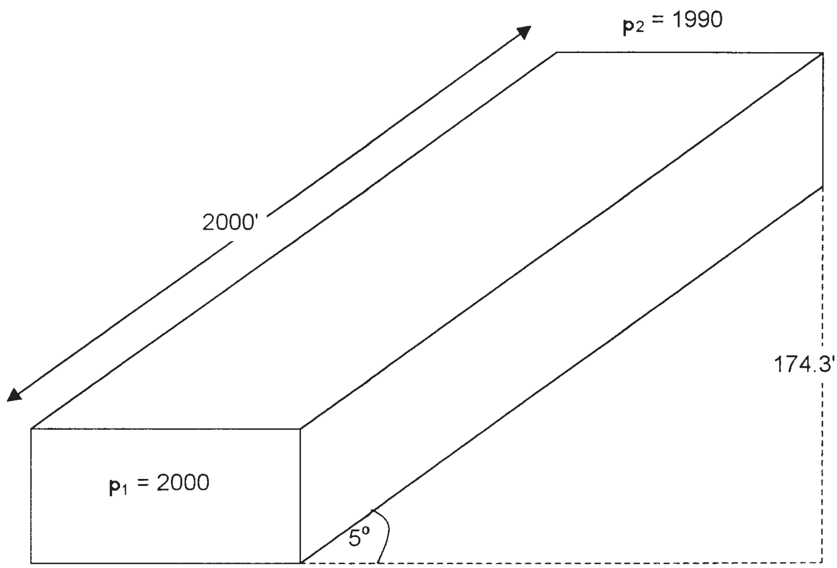



**Figure 6-12.** Example of a tilted layer.

- Notice, if we select Point 2 for the datum level, then

$$\Phi_1 = 2000 - \left(\frac{42}{144}\right)(174.3) = 1949.16\,\text{psi}$$

$$\Phi_2 = 1990 + \left(\frac{42}{144}\right)(0) = 1990\,\text{psi}$$

The above calculations indicate that regardless of the position of the datum level, the flow is downward from 2 to 1 with:

$$\Delta\Phi = 1990 - 1949.16 = 40.84 \text{ psi}$$

*Step 3.* Calculate the flow rate

$$q = \frac{(0.001127)(100)(6000)(40.84)}{(2)(2000)} = 6.9 \text{ bbl/day}$$

*Step 4.* Calculate the velocity:

$$\text{Apparent velocity} = \frac{(6.9)(5.615)}{6000} = 0.0065 \text{ ft/day}$$

$$\text{Actual velocity} = \frac{(6.9)(5.615)}{(0.15)(6000)} = 0.043 \text{ ft/day}$$

## Linear Flow of Slightly Compressible Fluids

Equation 6-6 describes the relationship that exists between pressure and volume for slightly compressible fluids, or:

$$V = V_{ref}\,[1 + c\,(p_{ref} - p)]$$

The above equation can be modified and written in terms of flow rate as:

$$q = q_{ref}\,[1 + c\,(p_{ref} - p)] \tag{6-18}$$

where $q_{ref}$ is the flow rate at some reference pressure $p_{ref}$. Substituting the above relationship in Darcy's equation gives:

$$\frac{q}{A} = \frac{q_{ref}\,[1 + c\,(p_{ref} - p)]}{A} = -0.001127\,\frac{k}{\mu}\,\frac{dp}{dx}$$

Separating the variables and arranging:

$$\frac{q_{ref}}{A}\int_{o}^{L} dx = -0.001127\,\frac{k}{\mu}\int_{p_1}^{p_2}\left[\frac{dp}{1 + c\,(p_{ref} - p)}\right]$$

Integrating gives:

$$q_{ref} = \left[\frac{0.001127\,kA}{\mu\,c\,L}\right]\ln\left[\frac{1 + c\,(p_{ref} - p_2)}{1 + c\,(p_{ref} - p_1)}\right] \tag{6-19}$$

where $q_{ref}$ = flow rate at a reference pressure $p_{ref}$, bbl/day
$p_1$ = upstream pressure, psi

$p_2$ = downstream pressure, psi
k = permeability, md
μ = viscosity, cp
c = average liquid compressibility, $psi^{-1}$

Selecting the upstream pressure $p_1$ as the reference pressure $p_{ref}$ and substituting in Equation 6-19 gives the flow rate at Point 1 as:

$$q_1 = \left[\frac{0.001127\ kA}{\mu cL}\right] \ln\left[1 + c\left(p_1 - p_2\right)\right] \tag{6-20}$$

Choosing the downstream pressure $p_2$ as the reference pressure and substituting in Equation 6-19 gives:

$$q_2 = \left[\frac{0.001127\ kA}{\mu cL}\right] \ln\left[\frac{1}{1 + c\left(p_2 - p_1\right)}\right] \tag{6-21}$$

where $q_1$ and $q_2$ are the flow rates at Points 1 and 2, respectively.

## Example 6-3

Consider the linear system given in Example 6-1 and, assuming a slightly compressible liquid, calculate the flow rate at both ends of the linear system. The liquid has an average compressibility of $21 \times 10^{-5}$ $psi^{-1}$.

## Solution

- Choosing the upstream pressure as the reference pressure gives:

$$q_1 = \left[\frac{(0.001127)(100)(6000)}{(2)(21 \times 10^{-5})(2000)}\right] \ln\left[1 + (21 \times 10^{-5})(2000 - 1990)\right]$$
$$= 1.689 \text{ bbl/day}$$

- Choosing the downstream pressure, gives:

$$q_2 = \left[\frac{(0.001127)(100)(6000)}{(2)(21 \times 10^{-5})(2000)}\right] \ln\left[\frac{1}{1 + (21 \times 10^{-5})(1990 - 2000)}\right]$$
$$= 1.692 \text{ bbl/day}$$

The above calculations show that $q_1$ and $q_2$ are not largely different, which is due to the fact that the liquid is slightly incompressible and its volume is not a strong function of pressure.

## Linear Flow of Compressible Fluids (Gases)

For a viscous (laminar) gas flow in a homogeneous-linear system, the real-gas equation-of-state can be applied to calculate the number of gas moles n at pressure p, temperature T, and volume V:

$$n = \frac{pV}{zRT}$$

At standard conditions, the volume occupied by the above n moles is given by:

$$V_{sc} = \frac{n\, z_{sc}\, R\, T_{sc}}{p_{sc}}$$

Combining the above two expressions and assuming $z_{sc} = 1$ gives:

$$\frac{pV}{zT} = \frac{p_{sc}\ V_{sc}}{T_{sc}}$$

Equivalently, the above relation can be expressed in terms of the flow rate as:

$$\frac{5.615pq}{zT} = \frac{p_{sc}\, Q_{sc}}{T_{sc}}$$

Rearranging:

$$\left(\frac{p_{sc}}{T_{sc}}\right)\left(\frac{zT}{p}\right)\left(\frac{Q_{sc}}{5.615}\right) = q \tag{6-22}$$

where $q$ = gas flow rate at pressure p in bbl/day
$Q_{sc}$ = gas flow rate at standard conditions, scf/day
$z$ = gas compressibility factor
$T_{sc}$, $p_{sc}$ = standard temperature and pressure in °R and psia, respectively

Replacing the gas flow rate q with that of Darcy's Law, i.e., Equation 6-12, gives:

$$\frac{q}{A} = \left(\frac{p_{sc}}{T_{sc}}\right)\left(\frac{zT}{p}\right)\left(\frac{Q_{sc}}{5.615}\right)\left(\frac{1}{A}\right) = -0.001127\frac{k}{\mu}\frac{dp}{dx}$$

The constant 0.001127 is to convert from Darcy's units to field units. Separating variables and arranging yields:

$$\left[\frac{q_{sc}\ p_{sc}\ T}{0.006328\ k\ T_{sc}\ A}\right]\int_0^L dx = -\int_{p_1}^{p_2}\frac{p}{z\mu_g}dp$$

Assuming constant z and $\mu_g$ over the specified pressures, i.e., $p_1$ and $p_2$, and integrating gives:

$$Q_{sc} = \frac{0.003164\ T_{sc}\ A\,k\ (p_1^2 - p_2^2)}{p_{sc}\ T\,L\,z\,\mu_g}$$

where $Q_{sc}$ = gas flow rate at standard conditions, scf/day
k = permeability, md
T = temperature, °R
$\mu_g$ = gas viscosity, cp
A = cross-sectional area, $ft^2$
L = total length of the linear system, ft

Setting $p_{sc}$ =14.7 psi and $T_{sc}$ = 520°R in the above expression gives:

$$Q_{sc} = \frac{0.111924\,A\,k\ (p_1^2 - p_2^2)}{T\,L\,z\,\mu_g} \tag{6-23}$$

It is essential to notice that those gas properties z and $\mu_g$ are a very strong function of pressure, but they have been removed from the integral to simplify the final form of the gas flow equation. The above equation is valid for applications when the pressure < 2,000 psi. The gas properties must be evaluated at the average pressure $\bar{p}$ as defined below.

$$\bar{p} = \sqrt{\frac{p_1^2 + p_2^2}{2}} \tag{6-24}$$

## Example 6-4

A linear porous media is flowing a 0.72 specific gravity gas at 120°F. The upstream and downstream pressures are 2,100 psi and 1,894.73 psi, respectively. The cross-sectional area is constant at 4,500 $ft^2$. The total length is 2,500 feet with an absolute permeability of 60 md. Calculate the gas flow rate in scf/day ($p_{sc}$ = 14.7 psia, $T_{sc}$ = 520°R).

## Solution

*Step 1.* Calculate average pressure by using Equation 6-24.

$$\bar{p} = \sqrt{\frac{2100^2 + 1894.73^2}{2}} = 2000\,\text{psi}$$

*Step 2.* Using the specific gravity of the gas, calculate its pseudo-critical properties by applying Equations 2-17 and 2-18.

$T_{pc} = 395.5°R \qquad p_{pc} = 668.4$ psia

*Step 3.* Calculate the pseudo-reduced pressure and temperature.

$$p_{pr} = \frac{2000}{668.4} = 2.99$$

$$T_{pr} = \frac{600}{395.5} = 1.52$$

*Step 4.* Determine the z-factor from the Standing-Katz chart (Figure 2-1) to give:

$z = 0.78$

*Step 5.* Solve for the viscosity of the gas by applying the Lee-Gonzalez-Eakin method (Equations 2-63 through 2-66) to give:

$\mu_g = 0.0173$ cp

*Step 6.* Calculate the gas flow rate by applying Equation 6-23.

$$Q_{sc} = \frac{(0.111924)(4500)(60)(2100^2 - 1894.73^2)}{(600)(0.78)(2500)(0.0173)}$$

$$= 1,224,242 \text{ scf/day}$$

## Radial Flow of Incompressible Fluids

In a radial flow system, all fluids move toward the producing well from all directions. Before flow can take place, however, a pressure differential must exist. Thus, if a well is to produce oil, which implies a flow of fluids through the formation to the wellbore, the pressure in the formation at the wellbore must be less than the pressure in the formation at some distance from the well.

The pressure in the formation at the wellbore of a producing well is known as the *bottom-hole flowing pressure* (flowing BHP, $p_{wf}$).

Consider Figure 6-13, which schematically illustrates the radial flow of an incompressible fluid toward a vertical well. The formation is considered to a uniform thickness h and a constant permeability k. Because the fluid is incompressible, the flow rate q must be constant at all radii. Due to the steady-state flowing condition, the pressure profile around the wellbore is maintained constant with time.

Let $p_{wf}$ represent the maintained bottom-hole flowing pressure at the wellbore radius $r_w$ and $p_e$ denote the external pressure at the external or drainage radius. Darcy's equation as described by Equation 6-13 can be used to determine the flow rate at any radius r:

$$v = \frac{q}{A_r} = 0.001127 \frac{k}{\mu} \frac{dp}{dr} \qquad (6\text{-}25)$$

where
- $v$ = apparent fluid velocity, bbl/day-ft$^2$
- $q$ = flow rate at radius r, bbl/day
- $k$ = permeability, md
- $\mu$ = viscosity, cp
- 0.001127 = conversion factor to express the equation in field units
- $A_r$ = cross-sectional area at radius r

The minus sign is no longer required for the radial system shown in Figure 6-13 as the radius increases in the same direction as the pressure. In other words, as the radius increases going away from the wellbore the

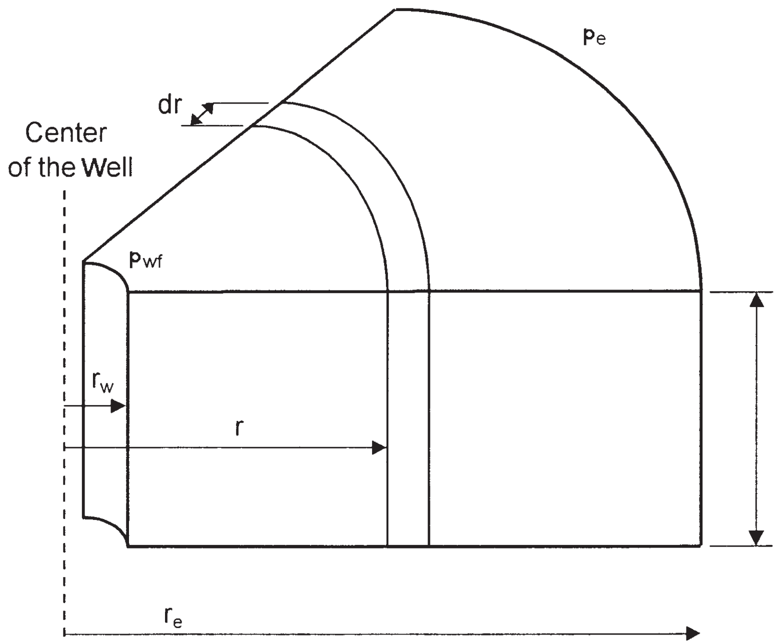



**Figure 6-13.** Radial flow model.

pressure also increases. At any point in the reservoir the cross-sectional area across which flow occurs will be the surface area of a cylinder, which is 2πrh, or:

$$v = \frac{q}{A_r} = \frac{q}{2\pi rh} = 0.001127 \frac{k}{\mu} \frac{dp}{dr}$$

The flow rate for a crude oil system is customarily expressed in surface units, i.e., stock-tank barrels (STB), rather than reservoir units. Using the symbol $Q_o$ to represent the oil flow as expressed in STB/day, then:

$$q = B_o Q_o$$

where $B_o$ is the oil formation volume factor bbl/STB. The flow rate in Darcy's equation can be expressed in STB/day to give:

$$\frac{Q_o B_o}{2\pi rh} = 0.001127 \frac{k}{\mu_o} \frac{dp}{dr}$$

Integrating the above equation between two radii, $r_1$ and $r_2$, when the pressures are $p_1$ and $p_2$ yields:

$$\int_{r_1}^{r_2}\left(\frac{Q_o}{2\pi h}\right)\frac{dr}{r} = 0.001127\int_{p_1}^{p_2}\left(\frac{k}{\mu_o B_o}\right)dp \tag{6-26}$$

For an incompressible system in a uniform formation, Equation 6-26 can be simplified to:

$$\frac{Q_o}{2\pi h}\int_{r_1}^{r_2}\frac{dr}{r} = \frac{0.001127\,k}{\mu_o\,B_o}\int_{p_1}^{p_2}dp$$

Performing the integration gives:

$$Q_o = \frac{0.00708\;k\,h\,(p_2 - p_1)}{\mu_o\;B_o\;\ln\;(r_2/r_1)}$$

Frequently the two radii of interest are the wellbore radius $r_w$ and the *external* or *drainage* radius $r_e$. Then:

$$Q_o = \frac{0.00708\,k\,h\,(p_e - p_w)}{\mu_o\,B_o\ln\,(r_e/r_w)} \tag{6-27}$$

where $Q_o$ = oil, flow rate, STB/day
$p_e$ = external pressure, psi
$p_{wf}$ = bottom-hole flowing pressure, psi
$k$ = permeability, md
$\mu_o$ = oil viscosity, cp
$B_o$ = oil formation volume factor, bbl/STB
$h$ = thickness, ft
$r_e$ = external or drainage radius, ft
$r_w$ = wellbore radius, ft

The external (drainage) radius $r_e$ is usually determined from the well spacing by equating the area of the well spacing with that of a circle, i.e.,

$$\pi\,r_e^2 = 43{,}560\;A$$

or

$$r_e = \sqrt{\frac{43{,}560\,A}{\pi}} \tag{6-28}$$

where A is the well spacing in acres.

In practice, neither the external radius nor the wellbore radius is generally known with precision. Fortunately, they enter the equation as a logarithm, so that the error in the equation will be less than the errors in the radii.

Equation 6-27 can be arranged to solve for the pressure p at any radius r to give:

$$p = p_{wf} + \left[\frac{Q_o B_o \mu_o}{0.00708\,kh}\right] \ln\left(\frac{r}{r_w}\right) \tag{6-29}$$

## Example 6-5

An oil well in the Nameless Field is producing at a stabilized rate of 600 STB/day at a stabilized bottom-hole flowing pressure of 1,800 psi. Analysis of the pressure buildup test data indicates that the pay zone is characterized by a permeability of 120 md and a uniform thickness of 25 ft. The well drains an area of approximately 40 acres. The following additional data are available:

$r_w = 0.25$ ft $\quad$ $A = 40$ acres
$B_o = 1.25$ bbl/STB $\quad$ $\mu_o = 2.5$ cp

Calculate the pressure profile (distribution) and list the pressure drop across 1 ft intervals from $r_w$ to 1.25 ft, 4 to 5 ft, 19 to 20 ft, 99 to 100 ft, and 744 to 745 ft.

## Solution

*Step 1.* Rearrange Equation 6-27 and solve for the pressure p at radius r.

$$p = p_{wf} + \left[\frac{\mu_o B_o Q_o}{0.00708\,k\,h}\right] \ln\,(r/r_w)$$

$$p = 1800 + \left[\frac{(2.5)\,(1.25)\,(600)}{(0.00708)\,(120)\,(25)}\right] \ln\left(\frac{r}{0.25}\right)$$

$$p = 1800 + 88.28\ \ln\left(\frac{r}{0.25}\right)$$

*Step 2.* Calculate the pressure at the designated radii.

| r, ft | p, psi | Radius Interval | Pressure drop |
|---|---|---|---|
| 0.25 | 1800 | | |
| 1.25 | 1942 | 0.25–1.25 | 1942 – 1800 = 142 psi |
| 4 | 2045 | | |
| 5 | 2064 | 4–5 | 2064 – 2045 = 19 psi |
| 19 | 2182 | | |
| 20 | 2186 | 19–20 | 2186 – 2182 = 4 psi |
| 99 | 2328 | | |
| 100 | 2329 | 99–100 | 2329 – 2328 = 1 psi |
| 744 | 2506.1 | | |
| 745 | 2506.2 | 744–745 | 2506.2 – 2506.1 = 0.1 psi |

Figure 6-14 shows the pressure profile on a function of radius for the calculated data.

Results of the above example reveal that the pressure drop just around the wellbore (i.e., 142 psi) is 7.5 times greater than at the 4–5 ft interval, 36 times greater than at 19–20 ft, and 142 times than that at the 99–100 ft interval. The reason for this large pressure drop around the wellbore is that the fluid is flowing in from a large drainage of 40 acres.

The external pressure pe used in Equation 6-27 cannot be measured readily, but $P_e$ does not deviate substantially from initial reservoir pressure if a strong and active aquifer is present.

Several authors have suggested that the average reservoir pressure $p_r$, which often is reported in well test results, should be used in performing material balance calculations and flow rate prediction. Craft and Hawkins (1959) showed that the average pressure is located at about 61% of the drainage radius $r_e$ for a steady-state flow condition. Substitute 0.61 $r_e$ in Equation 6-29 to give:

$$p(\text{at}\ r = 0.61\,r_e) = p_r = p_{wf} + \left[\frac{Q_o\,B_o\,\mu_o}{7.08\,k\,h}\right] \ln\left(\frac{0.61\,r_e}{r_w}\right)$$

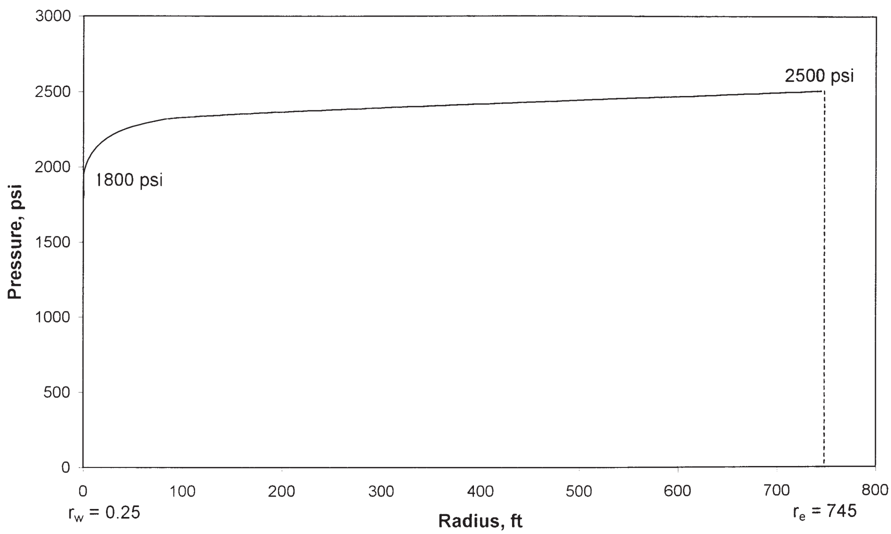



**Figure 6-14.** Pressure profile around the wellbore.

or in terms of flow rate:

$$Q_o = \frac{0.00708\,k\,h\,(p_r - p_{wf})}{\mu_o\,B_o\,\ln\left(\dfrac{0.61\,r_e}{r_w}\right)} \tag{6-30}$$

But, since $\ln\,(0.61r_e/r_w) = \ln\left(\dfrac{r_e}{r_w}\right) - 0.5$, then:

$$Q_o = \frac{0.00708\,k\,h\,(p_r - p_{wf})}{\mu_o\,B_o\left[\ln\left(\dfrac{r_e}{r_w}\right) - 0.5\right]} \tag{6-31}$$

Golan and Whitson (1986) suggest a method for approximating drainage area of wells producing from a common reservoir. The authors assume that the volume drained by a single well is proportional to its rate of flow. Assuming constant reservoir properties and a uniform thickness, the approximate drainage area of a single well, $A_w$, is:

$$A_w = A_T \left( \frac{q_w}{q_T} \right) \tag{6-32}$$

where $A_w$ = drainage area
$A_T$ = total area of the field
$q_T$ = total flow rate of the field
$q_w$ = well flow rate

## Radial Flow of Slightly Compressible Fluids

Craft et al. (1990) used Equation 6-18 to express the dependency of the flow rate on pressure for slightly compressible fluids. If this equation is substituted into the radial form of Darcy's Law, the following is obtained:

$$\frac{q}{A_r} = \frac{q_{ref}\,[1 + c\,(p_{ref} - p)]}{2\pi rh} = 0.001127\, \frac{k}{\mu}\, \frac{dp}{dr}$$

where $q_{ref}$ is the flow rate at some reference pressure $p_{ref}$.

Separating the variables in the above equation and integrating over the length of the porous medium gives:

$$\frac{q_{ref}\,\mu}{2\pi kh} \int_{r_w}^{r_e} \frac{dr}{r} = 0.001127 \int_{p_{wf}}^{p_e} \frac{dp}{1 + c\,(p_{ref} - p)}$$

or:

$$q_{ref} = \left[ \frac{0.00708\,kh}{\mu\, c\, \ln\left( \frac{r_e}{r_w} \right)} \right] \ln\left[ \frac{1 + c\left(p_e - p_{ref}\right)}{1 + c\left(p_{wf} - p_{ref}\right)} \right]$$

where $q_{ref}$ is oil flow rate at a reference pressure $p_{ref}$. Choosing the bottom-hole flow pressure $p_{wf}$ as the reference pressure and expressing the flow rate in STB/day gives:

$$Q_o = \left[ \frac{0.00708\,kh}{\mu_o B_o c_o \ln\left(\frac{r_e}{r_w}\right)} \right] \ln[1 + c_o (p_e - p_{wf})] \tag{6-33}$$

where $c_o$ = isothermal compressibility coefficient, $psi^{-1}$
$Q_o$ = oil flow rate, STB/day
k = permeability, md

## Example 6-6

The following data are available on a well in the Red River Field:

$p_e$ = 2506 psi  $p_{wf}$ = 1800
$r_e$ = 745′  $r_w$ = 0.25
$B_o$ = 1.25  $\mu_o$ = 2.5  $c_o = 25 \times 10^{-6}$ $psi^{-1}$
k = 0.12 Darcy  h = 25 ft.

Assuming a slightly compressible fluid, calculate the oil flow rate. Compare the result with that of incompressible fluid.

## Solution

For a slightly compressible fluid, the oil flow rate can be calculated by applying Equation 6-33:

$$Q_o = \left[ \frac{(0.00708)(120)(25)}{(2.5)(1.25)(25 \times 10^{-6}) \ln(745/0.25)} \right]$$
$$\times \ln\left[1 + (25 \times 10^{-6})(2506 - 1800)\right] = 595 \text{ STB/day}$$

Assuming an incompressible fluid, the flow rate can be estimated by applying Darcy's equation, i.e., Equation 6-27:

$$Q_o = \frac{(0.00708)(120)(25)(2506 - 1800)}{(2.5)(1.25)\ln(745/0.25)} = 600 \text{ STB/day}$$

## Radial Flow of Compressible Gases

The basic differential form of Darcy's Law for a horizontal laminar flow is valid for describing the flow of both gas and liquid systems. For a radial gas flow, the Darcy's equation takes the form:

$$q_{gr} = \frac{0.001127\,(2\pi rh)\,k}{\mu_g}\,\frac{dp}{dr} \tag{6-34}$$

where $q_{gr}$ = gas flow rate at radius r, bbl/day
r = radial distance, ft
h = zone thickness, ft
$\mu_g$ = gas viscosity, cp
p = pressure, psi
0.001127 = conversion constant from Darcy units to field units

The gas flow rate is usually expressed in scf/day. Referring to the gas flow rate at standard condition as $Q_g$, the gas flow rate $q_{gr}$ under pressure and temperature can be converted to that of standard condition by applying the real gas equation-of-state to both conditions, or

$$\frac{5.615\, q_{gr}\, p}{zRT} = \frac{Q_g\, p_{sc}}{z_{sc}\, R\, T_{sc}}$$

or

$$\left(\frac{p_{sc}}{5.615\, T_{sc}}\right)\left(\frac{zT}{p}\right) Q_g = q_{gr} \tag{6-35}$$

where $p_{sc}$ = standard pressure, psia
$T_{sc}$ = standard temperature, °R
$Q_g$ = gas flow rate, scf/day
$q_{gr}$ = gas flow rate at radius r, bbl/day
p = pressure at radius r, psia
T = reservoir temperature, °R
z = gas compressibility factor at p and T
$z_{sc}$ = gas compressibility factor at standard condition ≅ 1.0

Combining Equations 6-34 and 6-35 yields:

$$\left(\frac{p_{sc}}{5.615\ T_{sc}}\right)\left(\frac{zT}{p}\right)Q_g = \frac{0.001127(2\pi rh)k}{\mu_g}\ \frac{dp}{dr}$$

Assuming that $T_{sc} = 520°R$ and $p_{sc} = 14.7$ psia:

$$\left(\frac{T\ Q_g}{k\,h}\right)\frac{dr}{r} = 0.703\left(\frac{2p}{\mu_g\,z}\right)dp \qquad (6\text{-}36)$$

Integrating Equation 6-36 from the wellbore conditions ($r_w$ and $p_{wf}$) to any point in the reservoir (r and p) gives:

$$\int_{r_w}^{r}\left(\frac{T\ Q_g}{k\,h}\right)\frac{dr}{r} = 0.703\int_{p_{wf}}^{p}\left(\frac{2p}{\mu_g\,z}\right)dp \qquad (6\text{-}37)$$

Imposing Darcy's Law conditions on Equation 6-37, i.e.:

- **Steady-state flow**, which requires that $Q_g$ is constant at all radii
- **Homogeneous formation**, which implies that k and h are constant

gives:

$$\left(\frac{TQ_g}{k\,h}\right)\ln\left(\frac{r}{r_w}\right) = 0.703\int_{p_{wf}}^{p}\left(\frac{2p}{\mu_g\,z}\right)dp$$

The term $\int_{p_{wf}}^{p}\left(\frac{2p}{\mu_g\,z}\right)dp$ can be expanded to give:

$$\int_{p_{wf}}^{p}\left(\frac{2p}{\mu_g\,z}\right)dp = \int_{o}^{p}\left(\frac{2p}{\mu_g\,z}\right)dp - \int_{o}^{p_{wf}}\left(\frac{2p}{\mu_g\,z}\right)dp$$

Combining the above relationships yields:

$$\left(\frac{TQ_g}{kh}\right)\ln\left(\frac{r}{r_w}\right) = 0.703\left[\int_0^p\left(\frac{2p}{\mu_g z}\right)dp - \int_0^{p_{wf}}\left(\frac{2p}{\mu_g z}\right)dp\right] \tag{6-38}$$

The integral $\int_o^p 2p/(\mu_g\, z)\, dp$ is called the *real gas potential* or *real gas pseudopressure,* and it is usually represented by m(p) or $\psi$. Thus

$$m(p) = \psi = \int_o^p\left(\frac{2p}{\mu_g z}\right)dp \tag{6-39}$$

Equation 6-38 can be written in terms of the real gas potential to give:

$$\left(\frac{TQ_g}{kh}\right)\ln\frac{r}{r_w} = 0.703\,(\psi - \psi_w)$$

or

$$\psi = \psi_w + \frac{Q_g T}{0.703\ kh}\ln\frac{r}{r_w} \tag{6-40}$$

Equation 6-40 indicates that a graph of $\psi$ vs. ln $r/r_w$ yields a straight line of slope ($Q_gT/0.703kh$) and intercepts $\psi_w$ (Figure 6-15).

The flow rate is given exactly by

$$Q_g = \frac{0.703\ kh(\psi - \psi_w)}{T\ln\frac{r}{r_w}} \tag{6-41}$$

In the particular case when $r = r_e$, then:

$$Q_g = \frac{0.703\,kh\ (\psi_e - \psi_w)}{T\left(\ln\frac{r_e}{r_w}\right)} \tag{6-42}$$

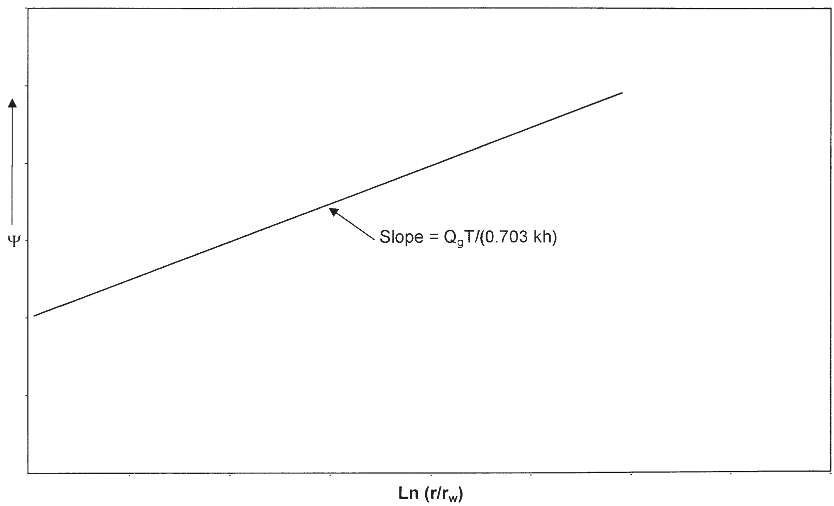



**Figure 6-15.** Graph of Ψ vs. ln $(r/r_w)$.

where $\psi_e$ = real gas potential as evaluated from 0 to $p_e$, $psi^2/cp$
$\psi_w$ = real gas potential as evaluated from 0 to $P_{wf}$, $psi^2/cp$
k = permeability, md
h = thickness, ft
$r_e$ = drainage radius, ft
$r_w$ = wellbore radius, ft
$Q_g$ = gas flow rate, scf/day

The gas flow rate is commonly expressed in Mscf/day, or

$$Q_g = \frac{kh(\psi_e - \psi_w)}{1422\ T\left(\ln\frac{r_e}{r_w}\right)} \tag{6-43}$$

where $Q_g$ = gas flow rate, Mscf/day.

Equation 6-43 can be expressed in terms of the average reservoir pressure $p_r$ instead of the initial reservoir pressure $p_e$ as:

$$Q_g = \frac{kh\ (\psi_r - \psi_w)}{1422\ T\left[\ln\left(\frac{r_e}{r_w}\right) - 0.5\right]} \tag{6-44}$$

To calculate the integral in Equation 6-43, the values of $2p/\mu_g z$ are calculated for several values of pressure p. Then $(2p/\mu_g z)$ versus p is plotted on a Cartesian scale and the area under the curve is calculated either numerically or graphically, where the area under the curve from p = 0 to any pressure p represents the value of $\psi$ corresponding to p. The following example will illustrate the procedure.

## Example 6-7

The following PVT data from a gas well in the Anaconda Gas Field is given below[1]:

| p (psi) | $\mu_g$(cp) | z |
|---|---|---|
| 0 | 0.0127 | 1.000 |
| 400 | 0.01286 | 0.937 |
| 800 | 0.01390 | 0.882 |
| 1200 | 0.01530 | 0.832 |
| 1600 | 0.01680 | 0.794 |
| 2000 | 0.01840 | 0.770 |
| 2400 | 0.02010 | 0.763 |
| 2800 | 0.02170 | 0.775 |
| 3200 | 0.02340 | 0.797 |
| 3600 | 0.02500 | 0.827 |
| 4000 | 0.02660 | 0.860 |
| 4400 | 0.02831 | 0.896 |

The well is producing at a stabilized bottom-hole flowing pressure of 3,600 psi. The wellbore radius is 0.3 ft. The following additional data are available:

$k = 65$ md  $h = 15$ ft  $T = 600°R$
$p_e = 4400$ psi  $r_e = 1000$ ft

Calculate the gas flow rate in Mscf/day.

## Solution

*Step 1.* Calculate the term $\left(\dfrac{2p}{\mu_g z}\right)$ for each pressure as shown below:

[1]Data from "Gas Well Testing, Theory, Practice & Regulations," Donohue and Ertekin, IHRDC Corporation (1982).

| p (psi) | $\mu_g$ (cp) | z | $\frac{2p}{\mu_g z}\left(\frac{psia}{cp}\right)$ |
|---|---|---|---|
| 0 | 0.0127 | 1.000 | 0 |
| 400 | 0.01286 | 0.937 | 66,391 |
| 800 | 0.01390 | 0.882 | 130,508 |
| 1200 | 0.01530 | 0.832 | 188,537 |
| 1600 | 0.01680 | 0.794 | 239,894 |
| 2000 | 0.01840 | 0.770 | 282,326 |
| 2400 | 0.02010 | 0.763 | 312,983 |
| 2800 | 0.02170 | 0.775 | 332,986 |
| 3200 | 0.02340 | 0.797 | 343,167 |
| 3600 | 0.02500 | 0.827 | 348,247 |
| 4000 | 0.02660 | 0.860 | 349,711 |
| 4400 | 0.02831 | 0.896 | 346,924 |

*Step 2.* Plot the term $\left(\frac{2p}{\mu_g z}\right)$ versus pressure as shown in Figure 6-16.

*Step 3.* Calculate numerically the area under the curve for each value of p. These areas correspond to the real gas potential ψ at each pressure. These ψ values are tabulated below (ψ versus p is also plotted in the figure).

| p (psi) | $\psi\left(\frac{psi^2}{cp}\right)$ |
|---|---|
| 400 | $13.2 \times 10^6$ |
| 800 | $52.0 \times 10^6$ |
| 1200 | $113.1 \times 10^6$ |
| 1600 | $198.0 \times 10^6$ |
| 2000 | $304.0 \times 10^6$ |
| 2400 | $422.0 \times 10^6$ |
| 2800 | $542.4 \times 10^6$ |
| 3200 | $678.0 \times 10^6$ |
| 3600 | $816.0 \times 10^6$ |
| 4000 | $950.0 \times 10^6$ |
| 4400 | $1089.0 \times 10^6$ |

*Step 4.* Calculate the flow rate by applying Equation 6-41.

$$p_w = 816.0 \times 10^6 \qquad p_e = 1089 \times 10^6$$

$$Q_g = \frac{(65)\,(15)\,(1089 - 816)\,10^6}{(1422)\,(600)\,\ln\,(1000/0.25)} = 37{,}614 \text{ Mscf/day}$$

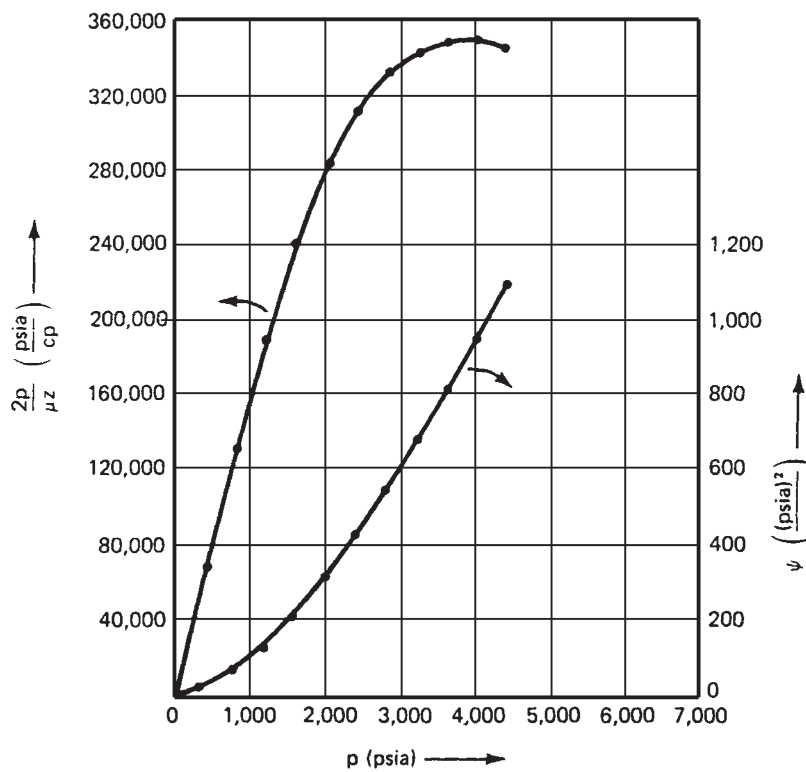



**Figure 6-16.** Real gas pseudopressure data for Example 6-7 (*After Donohue and Erekin, 1982*).

## Approximation of the Gas Flow Rate

The exact gas flow rate as expressed by the different forms of Darcy's Law, i.e., Equations 6-37 through 6-44, can be approximated by removing the term $\frac{2}{\mu_g z}$ outside the integral as a constant. It should be pointed out that the $z\mu_g$ is considered constant only under a pressure range of < 2,000 psi. Equation 6-43 can be rewritten as:

$$Q_g = \left[\frac{kh}{1422\,T\,\ln\left(\frac{r_e}{r_w}\right)}\right]\int_{p_{wf}}^{p_e}\left(\frac{2p}{\mu_g\,z}\right)dp$$

Removing the term and integrating gives:

$$Q_g = \frac{kh\,(p_e^2 - p_{wf}^2)}{1422\ T\ (\mu_g\ z)_{avg}\ \ln\left(\frac{r_e}{r_w}\right)} \tag{6-45}$$

where $Q_g$ = gas flow rate, Mscf/day
$k$ = permeability, md

The term $(\mu_g\ z)_{avg}$ is evaluated at an average pressure $\bar{p}$ that is defined by the following expression:

$$\bar{p} = \sqrt{\frac{p_{wf}^2 + p_e^2}{2}}$$

The above approximation method is called the *pressure-squared* method and is limited to flow calculations when the reservoir pressure is less that 2,000 psi. Other approximation methods are discussed in Chapter 7.

## Example 6-8

Using the data given in Example 6-7, re-solve for the gas flow rate by using the pressure-squared method. Compare with the exact method (i.e., real gas potential solution).

## Solution

*Step 1.* Calculate the arithmetic average pressure.

$$\bar{p} = \left[\frac{4400^2 + 3600^2}{2}\right]^{.5} = 4020\,\text{psi}$$

*Step 2.* Determine gas viscosity and gas compressibility factor at 4,020 psi.

$\mu_g = 0.0267$
$z = 0.862$

*Step 3.* Apply Equation 6-45:

$$Q_g = \frac{(65)(15)[4400^2 - 3600^2]}{(1422)(600)(0.0267)(0.862)\ln(1000/0.25)}$$
$$= 38{,}314 \text{ Mscf/day}$$

*Step 4.* Results show that the pressure-squared method approximates the exact solution of 37,614 with an absolute error of 1.86%. This error is due to the limited applicability of the pressure-squared method to a pressure range of < 2,000 psi.

## Horizontal Multiple-Phase Flow

When several fluid phases are flowing simultaneously in a horizontal porous system, the concept of the effective permeability to each phase and the associated physical properties must be used in Darcy's equation. For a radial system, the generalized form of Darcy's equation can be applied to each reservoir as follows:

$$q_o = 0.001127\left(\frac{2\pi rh}{\mu_o}\right)k_o\frac{dp}{dr}$$

$$q_w = 0.001127\left(\frac{2\pi rh}{\mu_w}\right)k_w\frac{dp}{dr}$$

$$q_g = 0.001127\left(\frac{2\pi rh}{\mu_g}\right)k_g\frac{dp}{dr}$$

where $k_o, k_w, k_g$ = effective permeability to oil, water, and gas, md
$\mu_o, \mu_w, \mu_g$ = viscosity to oil, water, and gas, cp
$q_o, q_w, q_g$ = flow rates for oil, water, and gas, bbl/day
$k$ = absolute permeability, md

The effective permeability can be expressed in terms of the relative and absolute permeability, as presented by Equations 5-1 through 5-2, to give:

$$k_o = k_{ro}\,k$$
$$k_w = k_{rw}\,k$$
$$k_g = k_{rg}\,k$$

Using the above concept in Darcy's equation and expressing the flow rate in standard conditions yield:

$$Q_o = 0.00708(rhk)\left(\frac{k_{ro}}{\mu_o \beta_o}\right)\frac{dp}{dr} \tag{6-46}$$

$$Q_w = 0.00708(rhk)\left(\frac{k_{rw}}{\mu_w \beta_w}\right)\frac{dp}{dr} \tag{6-47}$$

$$Q_g = 0.00708(rhk)\left(\frac{k_{rg}}{\mu_g \beta_g}\right)\frac{dp}{dr} \tag{6-48}$$

where $Q_o$, $Q_w$ = oil and water flow rates, STB/day
$B_o$, $B_w$ = oil and water formation volume factor, bbl/STB
$Q_g$ = gas flow rate, scf/day
$B_g$ = gas formation volume factor, bbl/scf
k = absolute permeability, md

The gas formation volume factor $B_g$ is previously expressed by Equation 2-54 as:

$$B_g = 0.005035\,\frac{zT}{p}, \text{ bbl/scf}$$

Performing the regular integration approach on Equations 6-46 through 6-48 yields:

- **Oil Phase**

$$Q_o = \frac{0.00708(kh)(k_{ro})(p_e - p_{wf})}{\mu_o B_o \ln(r_e/r_w)} \tag{6-49}$$

- **Water Phase**

$$Q_w = \frac{0.00708(kh)(k_{rw})(p_e - p_{wf})}{\mu_w B_w \ln(r_e/r_w)} \tag{6-50}$$

**• Gas Phase**

In terms of the real gas potential:

$$Q_g = \frac{(kh)\,k_{rg}\,(\psi_e - \psi_w)}{1422\,T\ln\,(r_e/r_w)} \tag{6-51}$$

In terms of the pressure-squared:

$$Q_g = \frac{(kh)\,k_{rg}\,(p_e^2 - p_{wf}^2)}{1422\,(\mu_g z)_{avg}\,T\ln\,(r_e/r_w)} \tag{6-52}$$

where $Q_g$ = gas flow rate, Mscf/day
k = absolute permeability, md
T = temperature, °R

In numerous petroleum engineering calculations, it is convenient to express the flow rate of any phase as a ratio of other flowing phase. Two important flow ratios are the "instantaneous" water-oil ratio (WOR) and "instantaneous" gas-oil ratio (GOR). The generalized form of Darcy's equation can be used to determine both flow ratios.

The water-oil ratio is defined as the ratio of the water flow rate to that of the oil. Both rates are expressed in stock-tank barrels per day, or:

$$WOR = \frac{Q_w}{Q_o}$$

Dividing Equation 6-46 by Equation 6-48 gives:

$$WOR = \left(\frac{k_{rw}}{k_{ro}}\right)\left(\frac{\mu_o\,B_o}{\mu_w\,B_w}\right) \tag{6-53}$$

where WOR = water-oil ratio, STB/STB.

The instantaneous GOR, as expressed in scf/STB, is defined as the *total* gas flow rate, i.e., free gas and solution gas, divided by the oil flow rate, or

$$GOR = \frac{Q_o\,R_s + Q_g}{Q_o}$$

or

$$GOR = R_s + \frac{Q_g}{Q_o} \qquad (6\text{-}54)$$

where GOR = "instantaneous" gas-oil ratio, scf/STB
$R_s$ = gas solubility, scf/STB
$Q_g$ = free gas flow rate, scf/day
$Q_o$ = oil flow rate, STB/day

Substituting Equations 6-46 and 6-48 into Equation 6-54 yields:

$$GOR = R_s + \left(\frac{k_{rg}}{k_{ro}}\right)\left(\frac{\mu_o B_o}{\mu_g B_g}\right) \qquad (6\text{-}55)$$

where $B_g$ is the gas formation volume factor as expressed in bbl/scf.

A complete discussion of the practical applications of the water-oil and gas-oil ratios is given in the subsequent chapters.

## UNSTEADY-STATE FLOW

Consider Figure 6-17A, which shows a shut-in well that is centered in a homogeneous circular reservoir of radius $r_e$ with a uniform pressure $p_i$ throughout the reservoir. This initial reservoir condition represents the zero producing time. If the well is allowed to flow at a constant flow rate of q, a pressure disturbance will be created at the sand face. The pressure at the wellbore, i.e., $p_{wf}$, will drop instantaneously as the well is opened. The pressure disturbance will move away from the wellbore at a rate that is determined by:

- Permeability
- Porosity
- Fluid viscosity
- Rock and fluid compressibilities

Section B in Figure 6-17 shows that at time $t_1$, the pressure disturbance has moved a distance $r_1$ into the reservoir. Notice that the pressure disturbance radius is continuously increasing with time. This radius is commonly called *radius of investigation* and referred to as $r_{inv}$. It is also

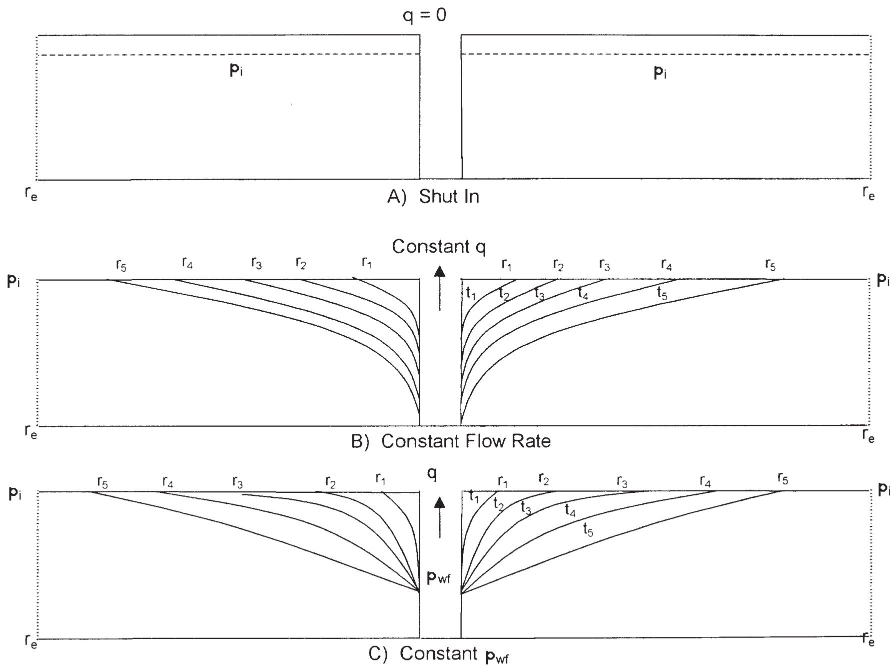



**Figure 6-17.** Pressure disturbance as a function of time.

important to point out that as long as the radius of investigation has not reached the reservoir boundary, i.e., $r_e$, the reservoir will be acting as if it is *infinite* in size. During this time we say that the reservoir is *infinite acting* because the outer drainage radius $r_e$ can be mathematically *infinite.*

A similar discussion to the above can be used to describe a well that is producing at a constant bottom-hole flowing pressure. Section C in Figure 6-17 schematically illustrates the propagation of the radius of investigation with respect to time. At time $t_4$, the pressure disturbance reaches the boundary, i.e., $r_{inv} = r_e$. This causes the pressure behavior to change.

Based on the above discussion, the transient (unsteady-state) flow is defined as **that time period during which the boundary has no effect on the pressure behavior in the reservoir and the reservoir will behave as its infinite in size.** Section B in Figure 6-17 shows that the transient flow period occurs during the time interval $0 < t < t_t$ for the constant flow rate scenario and during the time period $0 < t < t_4$ during the constant $p_{wf}$ scenario as depicted by Section C in Figure 6-17.

## Basic Transient Flow Equation

Under the steady-state flowing condition, the same quantity of fluid enters the flow system as leaves it. In the unsteady-state flow condition, the flow rate into an element of volume of a porous media may not be the same as the flow rate out of that element. Accordingly, the fluid content of the porous medium changes with time. The variables in unsteady-state flow additional to those already used for steady-state flow, therefore, become:

- Time, t
- Porosity, $\phi$
- Total compressibility, $c_t$

The mathematical formulation of the transient flow equation is based on combining three independent equations and a specifying set of boundary and initial conditions that constitute the unsteady-state equation. These equations and boundary conditions are briefly described below:

**a. Continuity Equation**
The continuity equation is essentially a material balance equation that accounts for every pound mass of fluid produced, injected, or remaining in the reservoir.

**b. Transport Equation**
The continuity equation is combined with the equation for fluid motion (transport equation) to describe the fluid flow rate "in" and "out" of the reservoir. Basically, the transport equation is Darcy's equation in its generalized differential form.

**c. Compressibility Equation**
The fluid compressibility equation (expressed in terms of density or volume) is used in formulating the unsteady-state equation with the objective of describing the changes in the fluid volume as a function of pressure.

**d. Initial and Boundary Conditions**
There are two boundary conditions and one initial condition required to complete the formulation and the solution of the transient flow equation. The two boundary conditions are:

- The formation produces at a constant rate into the wellbore.
- There is no flow across the outer boundary and the reservoir behaves as if it were infinite in size, i.e., $r_e = \infty$.

The initial condition simply states the reservoir is at a uniform pressure when production begins, i.e., time = 0.

Consider the flow element shown in Figure 6-18. The element has a width of dr and is located at a distance of r from the center of the well. The porous element has a differential volume of dV. According to the concept of the material-balance equation, the rate of mass flow into an element minus the rate of mass flow out of the element during a differential time Δt must be equal to the mass rate of accumulation during that time interval, or:

$$\begin{bmatrix} \text{mass entering} \\ \text{volume element} \\ \text{during interval } \Delta t \end{bmatrix} - \begin{bmatrix} \text{mass leaving} \\ \text{volume element} \\ \text{during interval } \Delta t \end{bmatrix} = \begin{bmatrix} \text{rate of mass} \\ \text{accumulation} \\ \text{during interval } \Delta t \end{bmatrix} \tag{6-56}$$

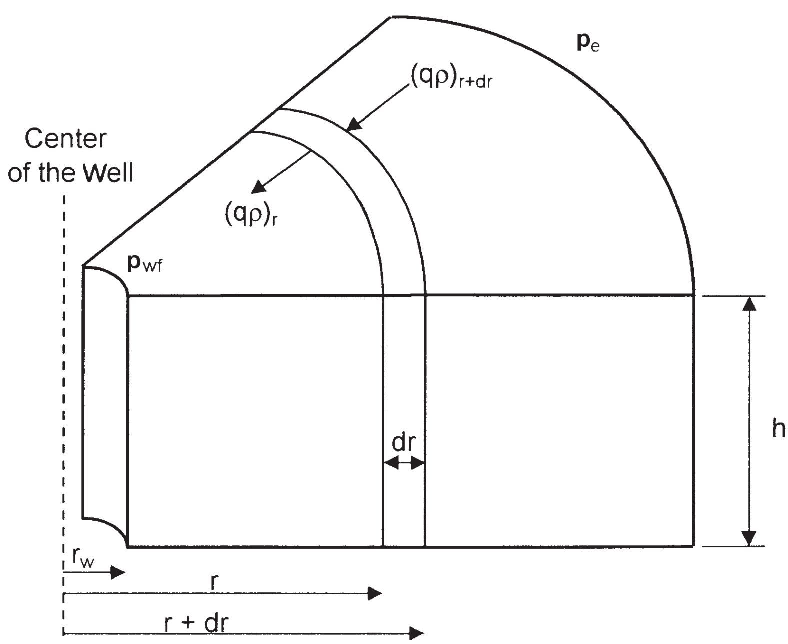



**Figure 6-18.** Illustration of radial flow.

The individual terms of Equation 6-56 are described below:

## Mass Entering the Volume Element During Time Interval Δt

$$(\text{Mass})_{in} = \Delta t \, [A \nu \rho]_{r+dr} \tag{6-57}$$

where $\nu$ = velocity of flowing fluid, ft/day
$\rho$ = fluid density at (r + dr), $lb/ft^3$
A = Area at (r + dr)
$\Delta t$ = time interval, days

The area of an element at the entering side is:

$$A_{r+dr} = 2\pi(r + dr)\, h \tag{6-58}$$

Combining Equation 6-58 with 6-47 gives:

$$[\text{Mass}]_{in} = 2\pi \, \Delta t \, (r + dr) \, h \, (\nu\rho)_{r+dr} \tag{6-59}$$

## Mass Leaving the Volume Element

Adopting the same approach as that of the leaving mass gives:

$$[\text{Mass}]_{out} = 2\pi \, \Delta t \, rh \, (\nu\rho)_r \tag{6-60}$$

## Total Accumulation of Mass

The volume of some element with a radius of r is given by:

$$V = \pi \, r^2 \, h$$

Differentiating the above equation with respect to r gives:

$$\frac{dV}{dr} = 2\pi rh$$

or:

$$dV = (2\pi rh)dr \tag{6-61}$$

Total mass accumulation during $\Delta t = dV \, [(\phi\rho)_{t + \Delta t} - (\phi\rho)_t]$

Substituting for dV yields:

Total mass accumulation = $(2\pi rh)\ dr\ [(\phi\rho)_{t+\Delta t} - (\phi\rho)_t]$ (6-62)

Replacing terms of Equation 6-56 with those of the calculated relationships gives:

$$2\pi h\ (r + dr)\ \Delta t\ (\phi\rho)_{r+dr} - 2\pi hr\ \Delta t\ (\phi\rho)_r = (2\pi rh)\ dr\ [(\phi\rho)_{t+\Delta t} - (\phi\rho)_t]$$

Dividing the above equation by (2πrh) dr and simplifying gives:

$$\frac{1}{(r)dr}[(r+dr)(\nu\rho)_{r+dr} - r(\nu\rho)_r] = \frac{1}{\Delta t}[(\phi\rho)_{t+\Delta t} - (\phi\rho)_t]$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}[r(\nu\rho)] = \frac{\partial}{\partial t}(\phi\rho) \tag{6-63}$$

where $\phi$ = porosity
$\rho$ = density, lb/ft$^3$
$\nu$ = fluid velocity, ft/day

Equation 6-63 is called the *continuity equation*, and it provides the principle of conservation of mass in radial coordinates.

The transport equation must be introduced into the continuity equation to relate the fluid velocity to the pressure gradient within the control volume dV. Darcy's Law is essentially the basic motion equation, which states that the velocity is proportional to the pressure gradient $(\partial p/\partial r)$. From Equation 6-25:

$$\nu = (5.615)(0.001127)\frac{k}{\mu}\frac{\partial p}{\partial r}$$

$$\nu = (0.006328)\frac{k}{\mu}\frac{\partial p}{\partial r} \tag{6-64}$$

where k = permeability, md
$\nu$ = velocity, ft/day

Combining Equation 6-64 with Equation 6-63 results in:

$$\frac{0.006328}{r}\frac{\partial}{\partial r}\left(\frac{k}{\mu}(\rho\, r)\frac{\partial p}{\partial r}\right)=\frac{\partial}{\partial t}(\phi\rho) \tag{6-65}$$

Expanding the right-hand side by taking the indicated derivatives eliminates the porosity from the partial derivative term on the right-hand side:

$$\frac{\partial}{\partial t}(\phi\rho)=\phi\frac{\partial\rho}{\partial t}+\rho\frac{\partial\phi}{\partial t} \tag{6-66}$$

As shown in Chapter 4, porosity is related to the formation compressibility by the following:

$$c_f=\frac{1}{\phi}\frac{\partial\phi}{\partial p} \tag{6-67}$$

Applying the chain rule of differentiation to $\partial\phi/\partial t$,

$$\frac{\partial\phi}{\partial t}=\frac{\partial\phi}{\partial p}\frac{\partial p}{\partial t}$$

Substituting Equation 6-67 into this equation,

$$\frac{\partial\phi}{\partial t}=\phi\, c_f\frac{\partial p}{\partial t}$$

Finally, substituting the above relation into Equation 6-66 and the result into Equation 6-65 gives:

$$\frac{0.006328}{r}\frac{\partial}{\partial r}\left(\frac{k}{\mu}(\rho r)\frac{\partial p}{\partial r}\right)=\rho\phi c_f\frac{\partial p}{\partial t}+\phi\frac{\partial\rho}{\partial t} \tag{6-68}$$

Equation 6-68 is the general partial differential equation used to describe the flow of any fluid flowing in a radial direction in porous media. In addition to the initial assumptions, Darcy's equation has been added, which implies that the flow is laminar. Otherwise, the equation is not restricted to any type of fluid and is equally valid for gases or liquids. Compressible and slightly compressible fluids, however, must be treated separately in order to develop practical equations that can be used to describe the flow behavior of these two fluids. The treatments of the following systems are discussed below:

- Radial flow of slightly compressible fluids
- Radial flow of compressible fluids

## Radial Flow of Slightly Compressible Fluids

To simplify Equation 6-68, assume that the permeability and viscosity are constant over pressure, time, and distance ranges. This leads to:

$$\left[\frac{0.006328\ k}{\mu\ r}\right]\frac{\partial}{\partial r}\left(r\rho\frac{\partial p}{\partial r}\right) = \rho\phi\, c_f\frac{\partial p}{\partial t} + \phi\frac{\partial \rho}{\partial t} \tag{6-69}$$

Expanding the above equation gives:

$$0.006328\left(\frac{k}{\mu}\right)\left[\frac{\rho}{r}\frac{\partial p}{\partial r} + \rho\frac{\partial^2 p}{\partial r^2} + \frac{\partial p}{\partial r}\frac{\partial \rho}{\partial r}\right] = \rho\phi\, c_f\left(\frac{\partial p}{\partial t}\right) + \phi\left(\frac{\partial \rho}{\partial t}\right)$$

Using the chain rule in the above relationship yields:

$$0.006328\left(\frac{k}{\mu}\right)\left[\frac{p}{r}\frac{\partial p}{\partial r} + p\frac{\partial^2 p}{\partial r^2} + \left(\frac{\partial \rho}{\partial \rho}\right)^2\frac{\partial \rho}{\partial \rho}\right] = p\phi c_f\left(\frac{\partial p}{\partial t}\right) + \phi\left(\frac{\partial p}{\partial t}\right)\left(\frac{\partial \rho}{\partial p}\right)$$

Dividing the above expression by the fluid density ρ gives

$$0.006328\left(\frac{k}{\mu}\right)\left[\frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial r^2} + \left(\frac{\partial p}{\partial r}\right)^2\left(\frac{1}{\rho}\frac{\partial \rho}{\partial \rho}\right)\right] = \phi\, c_f\left(\frac{\partial p}{\partial t}\right)$$

$$+\phi\frac{\partial p}{\partial t}\left(\frac{1}{\rho}\frac{\partial \rho}{\partial p}\right)$$

Recall that the compressibility of any fluid is related to its density by:

$$c = \frac{1}{\rho}\frac{\partial \rho}{\partial p}$$

Combining the above two equations gives:

$$0.006328\left(\frac{k}{\mu}\right)\left[\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+c\left[\frac{\partial p}{\partial r}\right]^2\right]=\phi\, c_f\left(\frac{\partial p}{\partial t}\right)+\phi\, c\left(\frac{\partial p}{\partial t}\right)$$

The term $c\left(\frac{\partial p}{\partial r}\right)^2$ is considered very small and may be ignored:

$$0.006328\left(\frac{k}{\mu}\right)\left[\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}\right]=\phi(c_f+c)\frac{\partial p}{\partial t} \tag{6-70}$$

Define total compressibility, $c_t$, as:

$$c_t = c + c_f \tag{6-71}$$

Combining Equations 6-69 with 6-70 and rearranging gives:

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}=\frac{\phi\mu c_t}{0.006328\,k}\frac{\partial p}{\partial t} \tag{6-72}$$

where the time t is expressed in days.

Equation 6-72 is called the *diffusivity equation.* It is one of the most important equations in petroleum engineering. The equation is particularly used in analysis well testing data where the time t is commonly recorded in hours. The equation can be rewritten as:

$$\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}=\frac{\phi\mu c_t}{0.000264\,k}\frac{\partial p}{\partial t} \tag{6-73}$$

where $k$ = permeability, md
$r$ = radial position, ft
$p$ = pressure, psia
$c_t$ = total compressibility, $psi^{-1}$
$t$ = time, hrs
$\phi$ = porosity, fraction
$\mu$ = viscosity, cp

When the reservoir contains more than one fluid, total compressibility should be computed as

$$c_t = c_o S_o + c_w S_w + c_g S_g + c_f \tag{6-74}$$

where $c_o$, $c_w$, and $c_g$ refer to the compressibility of oil, water, and gas, respectively, while $S_o$, $S_w$, and $S_g$ refer to the fractional saturation of these fluids. Note that the introduction of $c_t$ into Equation 6-72 does not make Equation 6-72 applicable to multiphase flow; the use of $c_t$, as defined by Equation 6-73, simply accounts for the compressibility of any *immobile* fluids that may be in the reservoir with the fluid that is flowing.

The term [$0.000264\ k/\phi\mu c_t$] (Equation 6-73) is called the diffusivity constant and is denoted by the symbol $\eta$, or:

$$\eta = \frac{0.000264\ k}{\phi\mu c_t} \tag{6-75}$$

The diffusivity equation can then be written in a more convenient form as:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = \frac{1}{\eta}\frac{\partial p}{\partial t} \tag{6-76}$$

The diffusivity equation as represented by Equation 6-76 is essentially designed to determine the pressure as a function of time t and position r.

Before discussing and presenting the different solutions to the diffusivity equation, it is necessary to summarize the assumptions and limitations used in developing Equation 6-76:

1. Homogeneous and isotropic porous medium
2. Uniform thickness
3. Single phase flow
4. Laminar flow
5. Rock and fluid properties independent of pressure

Notice that for a steady-state flow condition, the pressure at any point in the reservoir is constant and does not change with time, i.e., $\partial p/\partial t = 0$, and therefore Equation 6-76 reduces to:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = 0 \tag{6-77}$$

Equation 6-77 is called Laplace's equation for steady-state flow.

## Example 6-9

Show that the radial form of Darcy's equation is the solution to Equation 6-77.

## Solution

*Step 1.* Start with Darcy's Law as expressed by Equation 6-29

$$p = p_{wf} + \left[ \frac{Q_o B_o u_o}{0.00708\, k\, h} \right] \ln\left( \frac{r}{r_w} \right)$$

*Step 2.* For a steady-state incompressible flow, the term between the two brackets is constant and labeled as C, or:

$$p = p_{wf} + [C] \ln\left( \frac{r}{r_w} \right)$$

*Step 3.* Evaluate the above expression for the first and second derivative to give:

$$\frac{\partial p}{\partial r} = [C]\left( \frac{1}{r} \right)$$

$$\frac{\partial^2 p}{\partial r^2} = [C]\left( \frac{-1}{r^2} \right)$$

*Step 4.* Substitute the above two derivatives in Equation 6-77

$$\frac{-1}{r^2}[C] + \left( \frac{1}{r} \right)[C]\left( \frac{-1}{r} \right) = 0$$

*Step 5.* Results of Step 4 indicate that Darcy's equation satisfies Equation 6-77 and is indeed the solution to Laplace's equation.

To obtain a solution to the diffusivity equation (Equation 6-76), it is necessary to specify an initial condition and impose two boundary conditions. The initial condition simply states that the reservoir is at a uniform pressure $p_i$ when production begins. The two boundary conditions require that the well is producing at a constant production rate and that the reservoir behaves as if it were infinite in size, i.e., $r_e = \infty$.

Based on the boundary conditions imposed on Equation 6-76, there are two generalized solutions to the diffusivity equation:

- Constant-terminal-pressure solution
- Constant-terminal-rate solution

The **constant-terminal-pressure solution** is designed to provide the cumulative flow at any particular time for a reservoir in which the pressure at one boundary of the reservoir is held constant. This technique is frequently used in water influx calculations in gas and oil reservoirs.

The **constant-terminal-rate solution** of the radial diffusivity equation solves for the pressure change throughout the radial system providing that the flow rate is held constant at one terminal end of the radial system, i.e., at the producing well. These are two commonly used forms of the constant-terminal-rate solution:

- The $E_i$-function solution
- The dimensionless pressure $p_D$ solution

## CONSTANT-TERMINAL-PRESSURE SOLUTION

In the constant-rate solution to the radial diffusivity equation, the flow rate is considered to be constant at a certain radius (usually wellbore radius) and the pressure profile around that radius is determined as a function of time and position. In the constant-terminal-pressure solution, the pressure is known to be constant at some particular radius and the solution is designed to provide the cumulative fluid movement across the specified radius (boundary).

The constant-pressure solution is widely used in water influx calculations. A detailed description of the solution and its practical reservoir engineering applications is appropriately discussed in the water influx chapter of the book (Chapter 10).

## CONSTANT-TERMINAL-RATE SOLUTION

The constant-terminal-rate solution is an integral part of most transient test analysis techniques, such as with drawdown and pressure buildup analyses. Most of these tests involve producing the well at a **constant flow rate** and recording the flowing pressure as a function of time, i.e.,

$p(r_w,t)$. There are two commonly used forms of the constant-terminal-rate solution:

- The $E_i$-function solution
- The dimensionless pressure $p_D$ solution

These two popular forms of solution are discussed below.

## The $E_i$-Function Solution

Matthews and Russell (1967) proposed a solution to the diffusivity equation that is based on the following assumptions:

- Infinite acting reservoir, i.e., the reservoir is infinite in size
- The well is producing at a constant flow rate
- The reservoir is at a uniform pressure, $p_i$, when production begins
- The well, with a wellbore radius of $r_w$, is centered in a cylindrical reservoir of radius $r_e$
- No flow across the outer boundary, i.e., at $r_e$

Employing the above conditions, the authors presented their solution in the following form:

$$p(r,t) = p_i + \left[\frac{70.6\, Q_o\, \mu_o b_o}{kh}\right] E_i\left[\frac{-948\phi\mu_o c_t r^2}{kt}\right] \tag{6-78}$$

where $p(r,t)$ = pressure at radius r from the well after t hours
$t$ = time, hrs
$k$ = permeability, md
$Q_o$ = flow rate, STB/day

The mathematical function, $E_i$, is called the **exponential integral** and is defined by:

$$E_i(-x) = -\int_x^\infty \frac{e^{-u}\,du}{u} = \left[\ln x - \frac{x}{1!} + \frac{x^2}{2(2!)} - \frac{x^3}{3(3!)} + \text{etc.}\right] \tag{6-79}$$

Craft, Hawkins, and Terry (1991) presented the values of the $E_i$-function in tabulated and graphical forms as shown in Table 6-1 and Figure 6-19, respectively.

The $E_i$ solution, as expressed by Equation 6-78, is commonly referred to as the **line-source solution.** The exponential integral $E_i$ can be approximated by the following equation when its argument x is less than 0.01:

$$E_i(-x) = \ln(1.781x) \tag{6-80}$$

where the argument x in this case is given by:

$$x = \frac{948\phi\mu\, c_t\, r^2}{k\,t}$$

Equation 6-80 approximates the $E_i$-function with less than 0.25% error. Another expression that can be used to approximate the $E_i$-function for the range $0.01 < x < 3.0$ is given by:

$$E_i(-x) = a_1 + a_2 \ln(x) + a_3 [\ln(x)]^2 + a_4 [\ln(x)]^3 + a_5 x + a_6 x^2 + a_7 x^3 + a_8 / x \tag{6-81}$$

With the coefficients $a_1$ through $a_8$ having the following values:

$a_1 = -0.33153973$ $\quad a_2 = -0.81512322$ $\quad a_3 = 5.22123384(10^{-2})$
$a_4 = 5.9849819(10^{-3})$ $\quad a_5 = 0.662318450$ $\quad a_6 = -0.12333524$
$a_7 = 1.0832566(10^{-2})$ $\quad a_8 = 8.6709776(10^{-4})$

The above relationship approximated the $E_i$-values with an average error of 0.5%.

It should be pointed out that for $x > 10.9$, the $E_i(-x)$ can be considered zero for all practical reservoir engineering calculations.

## Example 6-10

An oil well is producing at a constant flow rate of 300 STB/day under unsteady-state flow conditions. The reservoir has the following rock and fluid properties:

$B_o = 1.25$ bbl/STB $\quad \mu_o = 1.5$ cp $\quad c_t = 12 \times 10^{-6}$ psi$^{-1}$
$k_o = 60$ md $\quad h = 15$ ft $\quad p_i = 4000$ psi
$\phi = 15\%$ $\quad r_w = 0.25$ ft

**Table 6-1**
**Values of the $-E_i(-x)$ as a Function of x**
**(*After Craft, Hawkins, and Terry, 1991*)**

| x | $-E_i(-x)$ | x | $-E_i(-x)$ | x | $-E_i(-x)$ |
|---|---|---|---|---|---|
| 0.1 | 1.82292 | 4.3 | 0.00263 | 8.5 | 0.00002 |
| 0.2 | 1.22265 | 4.4 | 0.00234 | 8.6 | 0.00002 |
| 0.3 | 0.90568 | 4.5 | 0.00207 | 8.7 | 0.00002 |
| 0.4 | 0.70238 | 4.6 | 0.00184 | 8.8 | 0.00002 |
| 0.5 | 0.55977 | 4.7 | 0.00164 | 8.9 | 0.00001 |
| 0.6 | 0.45438 | 4.8 | 0.00145 | 9.0 | 0.00001 |
| 0.7 | 0.37377 | 4.9 | 0.00129 | 9.1 | 0.00001 |
| 0.8 | 0.31060 | 5.0 | 0.00115 | 9.2 | 0.00001 |
| 0.9 | 0.26018 | 5.1 | 0.00102 | 9.3 | 0.00001 |
| 1.0 | 0.21938 | 5.2 | 0.00091 | 9.4 | 0.00001 |
| 1.1 | 0.18599 | 5.3 | 0.00081 | 9.5 | 0.00001 |
| 1.2 | 0.15841 | 5.4 | 0.00072 | 9.6 | 0.00001 |
| 1.3 | 0.13545 | 5.5 | 0.00064 | 9.7 | 0.00001 |
| 1.4 | 0.11622 | 5.6 | 0.00057 | 9.8 | 0.00001 |
| 1.5 | 0.10002 | 5.7 | 0.00051 | 9.9 | 0.00000 |
| 1.6 | 0.08631 | 5.8 | 0.00045 | 10.0 | 0.00000 |
| 1.7 | 0.07465 | 5.9 | 0.00040 | | |
| 1.8 | 0.06471 | 6.0 | 0.00036 | | |
| 1.9 | 0.05620 | 6.1 | 0.00032 | | |
| 2.0 | 0.04890 | 6.2 | 0.00029 | | |
| 2.1 | 0.04261 | 6.3 | 0.00026 | | |
| 2.2 | 0.03719 | 6.4 | 0.00023 | | |
| 2.3 | 0.03250 | 6.5 | 0.00020 | | |
| 2.4 | 0.02844 | 6.6 | 0.00018 | | |
| 2.5 | 0.02491 | 6.7 | 0.00016 | | |
| 2.6 | 0.02185 | 6.8 | 0.00014 | | |
| 2.7 | 0.01918 | 6.9 | 0.00013 | | |
| 2.8 | 0.01686 | 7.0 | 0.00012 | | |
| 2.9 | 0.01482 | 7.1 | 0.00010 | | |
| 3.0 | 0.01305 | 7.2 | 0.00009 | | |
| 3.1 | 0.01149 | 7.3 | 0.00008 | | |
| 3.2 | 0.01013 | 7.4 | 0.00007 | | |
| 3.3 | 0.00894 | 7.5 | 0.00007 | | |
| 3.4 | 0.00789 | 7.6 | 0.00006 | | |
| 3.5 | 0.00697 | 7.7 | 0.00005 | | |
| 3.6 | 0.00616 | 7.8 | 0.00005 | | |
| 3.7 | 0.00545 | 7.9 | 0.00004 | | |
| 3.8 | 0.00482 | 8.0 | 0.00004 | | |
| 3.9 | 0.00427 | 8.1 | 0.00003 | | |
| 4.0 | 0.00378 | 8.2 | 0.00003 | | |
| 4.1 | 0.00335 | 8.3 | 0.00003 | | |
| 4.2 | 0.00297 | 8.4 | 0.00002 | | |

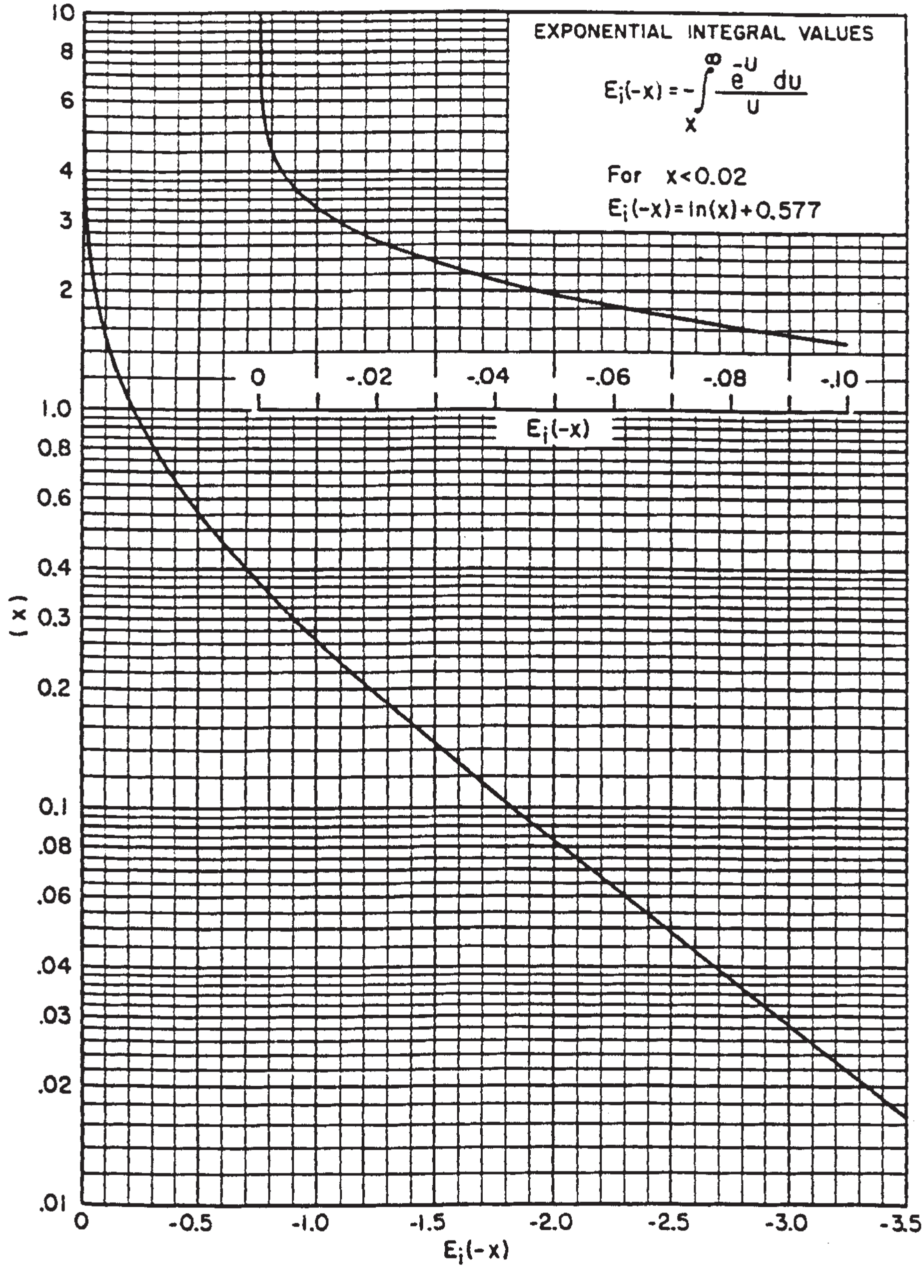



**Figure 6-19.** The $E_i$-function. (*After Craft, Hawkins, and Terry, 1991.*)

1. Calculate pressure at radii of 0.25, 5, 10, 50, 100, 500, 1,000, 1,500, 2,000, and 2,500 feet, for 1 hour.
   Plot the results as:

   a. Pressure versus logarithm of radius
   b. Pressure versus radius

2. Repeat part 1 for t = 12 hours and 24 hours. Plot the results as pressure versus logarithm of radius.

## Solution

*Step 1.* From Equation 6-78:

$$p(r,t) = 4000 + \left[\frac{70.6\,(300)\,(1.5)\,(1.25)}{(60)\,(15)}\right]$$

$$\times\, E_i\left[\frac{-948\,(.15)(1.5)\,(12\times 10^{-6})\,r^2}{(60)\,(t)}\right]$$

$$p(r,t) = 4000 + 44.125\,E_i\left[-42.6(10^{-6})\,\frac{r^2}{t}\right]$$

*Step 2.* Perform the required calculations after one hour in the following tabulated form:

**Elapsed Time t = 1 hr**

| r, ft | $x = -42.6(10^{-6})\,\frac{r^2}{1}$ | $E_i\,(-x)$ | $p(r,1) = 4000 + 44.125\,E_i\,(-x)$ |
|---|---|---|---|
| 0.25 | $-2.6625(10^{-6})$ | −12.26* | 3459 |
| 5 | −0.001065 | −6.27* | 3723 |
| 10 | −0.00426 | −4.88* | 3785 |
| 50 | −0.1065 | −1.76† | 3922 |
| 100 | −0.4260 | −0.75† | 3967 |
| 500 | −10.65 | 0 | 4000 |
| 1000 | −42.60 | 0 | 4000 |
| 1500 | −95.85 | 0 | 4000 |
| 2000 | −175.40 | 0 | 4000 |
| 2500 | −266.25 | 0 | 4000 |

**As calculated from Equation 6-29*

*†From Figure 6-19*

*Step 3.* Show results of the calculation graphically as illustrated in Figures 6-20 and 6-21.

*Step 4.* Repeat the calculation for t = 12 and 24 hrs.

**Elapsed Time t = 12 hrs**

| r, ft | $x = 42.6(10^{-6})\ \frac{r^2}{12}$ | $E_i\ (-x)$ | $p(r,12) = 4000 + 44.125\ E_i\ (-x)$ |
|---|---|---|---|
| 0.25 | $0.222\ (10^{-6})$ | −14.74* | 3350 |
| 5 | $88.75\ (10^{-6})$ | −8.75* | 3614 |
| 10 | $355.0\ (10^{-6})$ | −7.37* | 3675 |
| 50 | 0.0089 | −4.14* | 3817 |
| 100 | 0.0355 | −2.81† | 3876 |
| 500 | 0.888 | −0.269 | 3988 |
| 1000 | 3.55 | −0.0069 | 4000 |
| 1500 | 7.99 | $-3.77(10^{-5})$ | 4000 |
| 2000 | 14.62 | 0 | 4000 |
| 2500 | 208.3 | 0 | 4000 |

**As calculated from Equation 6-29*
*†From Figure 6-19*

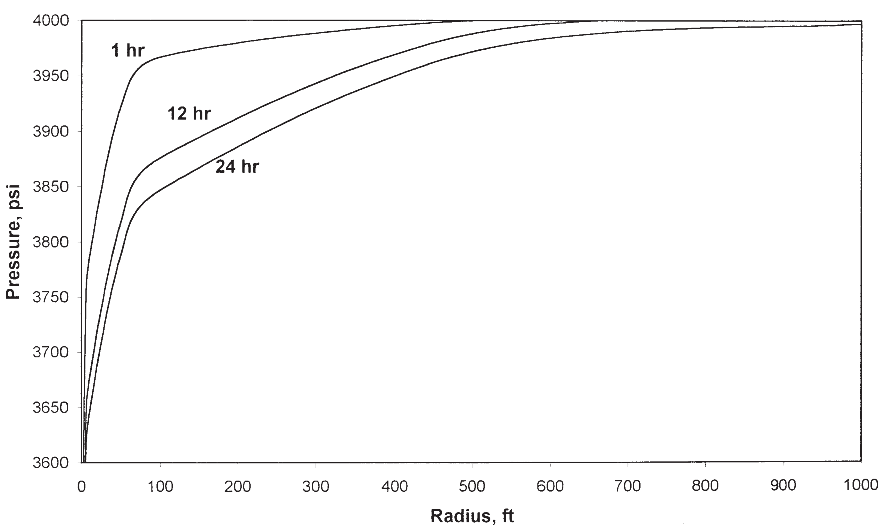



**Figure 6-20.** Pressure profiles as a function of time.

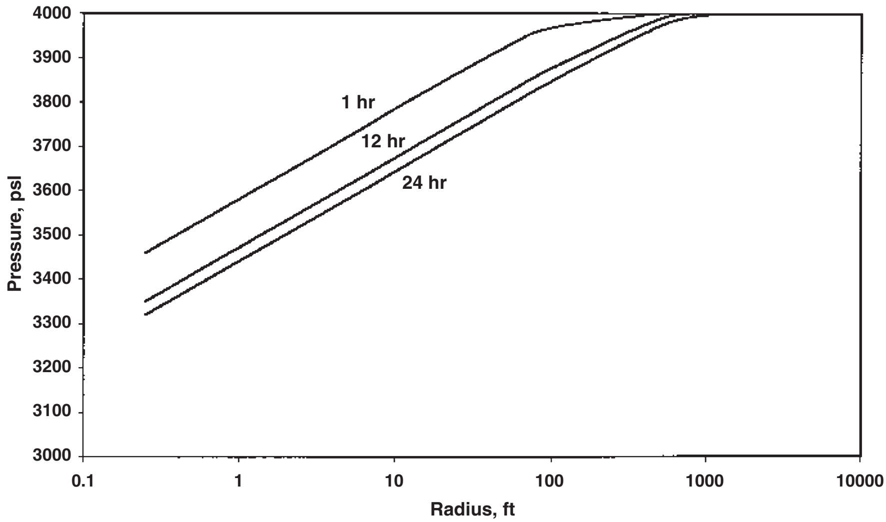



**Figure 6-21.** Pressure profiles as a function of time on a semi log scale.

**Elapsed Time t = 24 hrs**

| r, ft | $x = 42.6(10^{-6})\ \dfrac{r^2}{24}$ | $E_i\ (-x)$ | $p(r,24) = 4000 + 44.125\ E_i\ (-x)$ |
|---|---|---|---|
| 0.25 | $-0.111\ (10^{-6})$ | −15.44* | 3319 |
| 5 | $-44.38\ (10^{-6})$ | −9.45* | 3583 |
| 10 | $-177.5\ (10^{-6})$ | −8.06* | 3644 |
| 50 | −0.0045 | −4.83* | 3787 |
| 100 | −0.0178 | −3.458† | 3847 |
| 500 | −0.444 | −0.640 | 3972 |
| 1000 | −1.775 | −0.067 | 3997 |
| 1500 | −3.995 | −0.0427 | 3998 |
| 2000 | −7.310 | $8.24\ (10^{-6})$ | 4000 |
| 2500 | −104.15 | 0 | 4000 |

**As calculated from Equation 6-29*
*†From Figure 6-19*

*Step 5.* Results of Step 4 are shown graphically in Figure 6-21.

The above example shows that most of the pressure loss occurs close to the wellbore; accordingly, near-wellbore conditions will exert the greatest influence on flow behavior. Figure 6-21 shows that the pressure profile and the drainage radius are continuously changing with time.

When the parameter x in the $E_i$-function is less than 0.01, the log approximation as expressed by Equation 6-80 can be used in Equation 6-78 to give:

$$p(r,t) = p_i - \frac{162.6 Q_o B_o m_o}{kh}\left[\log\left(\frac{k t}{\phi \mu_o c_t r^2}\right) - 3.23\right] \tag{6-82}$$

For most of the transient flow calculations, engineers are primarily concerned with the behavior of the bottom-hole flowing pressure at the wellbore, i.e., $r = r_w$. Equation 6-82 can be applied at $r = r_w$ to yield:

$$p_{wf} = p_i - \frac{162.6 Q_o B_o m_o}{k h}\left[\log\left(\frac{k t}{\phi \mu_o c_t r_w^2}\right) - 3.23\right] \tag{6-83}$$

where $k$ = permeability, md
$t$ = time, hr
$c_t$ = total compressibility, $psi^{-1}$

It should be noted that Equations 6-82 and 6-83 cannot be used until the flow time t exceeds the limit imposed by the following constraint:

$$t > 9.48 \times 10^4 \frac{\phi \mu_o c_t r^2}{k} \tag{6-84}$$

where $t$ = time, hr
$k$ = permeability, md

## Example 6-11

Using the data in Example 6-10, estimate the bottom-hole flowing pressure after 10 hours of production.

## Solution

*Step 1.* Equation 6-83 can be used to calculate $p_{wf}$ only if the time exceeds the time limit imposed by Equation 6-84, or:

$$t = 9.48(10^4)\,\frac{(0.15)(1.5)(12\times10^{-6})(0.25)^2}{60} = 0.000267\ \text{hr}$$
$$= 0.153\ \text{sec}$$

For all practical purposes, Equation 6-83 can be used anytime during the transient flow period to estimate the bottom-hole pressure.

*Step 2.* Since the specified time of 10 hr is greater than 0.000267 hr, the $p_{wf}$ can be estimated by applying Equation 6-83.

$$p_{wf} = 4000 - \frac{162.6(300)(1.25)(1.5)}{(60)(15)}$$
$$\times\left[\log\left(\frac{(60)(10)}{(0.15)(1.5)(12\times10^{-6})(0.25)^2}\right) - 3.23\right] = 3358\ \text{psi}$$

The second form of solution to the diffusivity equation is called the *dimensionless pressure drop* and is discussed below.

## The Dimensionless Pressure Drop ($p_D$) Solution

Well test analysis often makes use of the concept of the dimensionless variables in solving the unsteady-state flow equation. The importance of dimensionless variables is that they simplify the diffusivity equation and its solution by combining the reservoir parameters (such as permeability, porosity, etc.) and thereby reduce the total number of unknowns.

To introduce the concept of the dimensionless pressure drop solution, consider for example Darcy's equation in a radial form as given previously by Equation 6-27.

$$Q_o = 0.00708\,\frac{kh\left(p_e - p_{wf}\right)}{\mu_o\, B_o \ln\left(r_e / r_w\right)}$$

Rearrange the above equation to give:

$$\frac{p_e - p_{wf}}{\left(\dfrac{Q_o\, B_o\, \mu_o}{0.00708\ kh}\right)} = \ln\left(\frac{r_e}{r_w}\right) \qquad (6\text{-}85)$$

It is obvious that the right-hand side of the above equation has no units (i.e., dimensionless) and, accordingly, the left-hand side must be dimensionless. Since the left-hand side is dimensionless, and ($p_e - p_{wf}$) has the units of psi, it follows that the term [$Q_o$ $B_o$ $\mu_o$/(0.00708kh)] has units of pressure. In fact, any pressure difference divided by [$Q_o$ $B_o$ $\mu_o$/(0.00708kh)] is a dimensionless pressure. Therefore, Equation 6-85 can be written in a dimensionless form as:

$$p_D = \ln(r_{eD})$$

where

$$p_D = \frac{p_e - p_{wf}}{\left(\dfrac{Q_o B_o \mu_o}{0.00708\,k\,h}\right)}$$

This concept can be extended to consider unsteady-state equations where the time is a variable. Defining:

$$r_{eD} = \frac{r_e}{r_w}$$

In transient flow analysis, the dimensionless pressure $p_D$ is always a function of dimensionless time that is defined by the following expression:

$$p_D = \frac{p_i - p(r,t)}{\left(\dfrac{Q_o\ B_o\ \mu_o}{0.00708\,k\,h}\right)} \tag{6-86}$$

In transient flow analysis, the dimensionless pressure $p_D$ is always a function of dimensionless time that is defined by the following expression:

$$t_D = \frac{0.000264\,kt}{\phi \mu c_t\, r_w^2} \tag{6-87}$$

The above expression is only one form of the dimensionless time. Another definition in common usage is $t_{DA}$, the dimensionless time based on total drainage area.

$$t_{DA} = \frac{0.000264\,kt}{\phi\mu c_t\,A} = t_D\left(\frac{r_w^2}{A}\right) \tag{6-87a}$$

where $A$ = total drainage area = $\pi\,r_e^2$
$r_e$ = drainage radius, ft
$r_w$ = wellbore radius, ft

The dimensionless pressure $p_D$ also varies with location in the reservoir as represented by the dimensionless radial distances $r_D$ and $r_{eD}$ that are defined by:

$$r_D = \frac{r}{r_w} \tag{6-88}$$

and

$$r_{eD} = \frac{r_e}{r_w} \tag{6-89}$$

where $p_D$ = dimensionless pressure drop
$r_{eD}$ = dimensionless external radius
$t_D$ = dimensionless time
$r_D$ = dimensionless radius
$t$ = time, hr
$p(r,t)$ = pressure at radius r and time t
$k$ = permeability, md
$\mu$ = viscosity, cp

The above dimensionless groups (i.e., $p_D$, $t_D$, and $r_D$) can be introduced into the diffusivity equation (Equation 6-76) to transform the equation into the following dimensionless form:

$$\frac{\partial^2 p_D}{\partial r_D^2} + \frac{1}{r_D}\frac{\partial p_D}{\partial r_D} = \frac{\partial p_D}{\partial t_D} \tag{6-90}$$

Van Everdingen and Hurst (1949) proposed an analytical solution to the above equation by assuming:

- Perfectly radial reservoir system
- The producing well is in the center and producing at a constant production rate of Q
- Uniform pressure $p_i$ throughout the reservoir before production
- No flow across the external radius $r_e$

Van Everdingen and Hurst presented the solution to Equation 6-89 in a form of infinite series of exponential terms and Bessel functions. The authors evaluated this series for several values of $r_{eD}$ over a wide range of values for $t_D$. Chatas (1953) and Lee (1982) conveniently tabulated these solutions for the following two cases:

- Infinite-acting reservoir
- Finite-radial reservoir

## Infinite-Acting Reservoir

When a well is put on production at a constant flow rate after a shut-in period, the pressure in the wellbore begins to drop and causes a pressure disturbance to spread in the reservoir. The influence of the reservoir boundaries or the shape of the drainage area does not affect the rate at which the pressure disturbance spreads in the formation. That is why the transient state flow is also called the *infinite acting state.* During the infinite acting period, the declining rate of wellbore pressure and the manner by which the pressure disturbance spreads through the reservoir are determined by reservoir and fluid characteristics such as:

- Porosity, $\phi$
- Permeability, k
- Total compressibility, $c_t$
- Viscosity, $\mu$

For an infinite-acting reservoir, i.e., $r_{eD} = \infty$, the dimensionless pressure drop function $p_D$ is strictly a function of the dimensionless time $t_D$, or:

$$p_D = f(t_D)$$

Chatas and Lee tabulated the $p_D$ values for the infinite-acting reservoir as shown in Table 6-2. The following mathematical expressions can be used to approximate these tabulated values of $p_D$:

**Table 6-2**
**$p_D$ vs. $t_D$—Infinite-Radial System, Constant-Rate at the Inner Boundary (*After Lee, J.,* Well Testing, *SPE Textbook Series.*) (*Permission to publish by the SPE, copyright SPE, 1982*)**

| $t_D$ | $p_D$ | $t_D$ | $p_D$ | $t_D$ | $p_D$ |
|---|---|---|---|---|---|
| 0 | 0 | 0.15 | 0.3750 | 60.0 | 2.4758 |
| 0.0005 | 0.0250 | 0.2 | 0.4241 | 70.0 | 2.5501 |
| 0.001 | 0.0352 | 0.3 | 0.5024 | 80.0 | 2.6147 |
| 0.002 | 0.0495 | 0.4 | 0.5645 | 90.0 | 2.6718 |
| 0.003 | 0.0603 | 0.5 | 0.6167 | 100.0 | 2.7233 |
| 0.004 | 0.0694 | 0.6 | 0.6622 | 150.0 | 2.9212 |
| 0.005 | 0.0774 | 0.7 | 0.7024 | 200.0 | 3.0636 |
| 0.006 | 0.0845 | 0.8 | 0.7387 | 250.0 | 3.1726 |
| 0.007 | 0.0911 | 0.9 | 0.7716 | 300.0 | 3.2630 |
| 0.008 | 0.0971 | 1.0 | 0.8019 | 350.0 | 3.3394 |
| 0.009 | 0.1028 | 1.2 | 0.8672 | 400.0 | 3.4057 |
| 0.01 | 0.1081 | 1.4 | 0.9160 | 450.0 | 3.4641 |
| 0.015 | 0.1312 | 2.0 | 1.0195 | 500.0 | 3.5164 |
| 0.02 | 0.1503 | 3.0 | 1.1665 | 550.0 | 3.5643 |
| 0.025 | 0.1669 | 4.0 | 1.2750 | 600.0 | 3.6076 |
| 0.03 | 0.1818 | 5.0 | 1.3625 | 650.0 | 3.6476 |
| 0.04 | 0.2077 | 6.0 | 1.4362 | 700.0 | 3.6842 |
| 0.05 | 0.2301 | 7.0 | 1.4997 | 750.0 | 3.7184 |
| 0.06 | 0.2500 | 8.0 | 1.5557 | 800.0 | 3.7505 |
| 0.07 | 0.2680 | 9.0 | 1.6057 | 850.0 | 3.7805 |
| 0.08 | 0.2845 | 10.0 | 1.6509 | 900.0 | 3.8088 |
| 0.09 | 0.2999 | 15.0 | 1.8294 | 950.0 | 3.8355 |
| 0.1 | 0.3144 | 20.0 | 1.9601 | 1,000.0 | 3.8584 |
| | | 30.0 | 2.1470 | | |
| | | 40.0 | 2.2824 | | |
| | | 50.0 | 2.3884 | | |

*Notes: For $t_D < 0.01$, $p_D \cong 2\ \text{Z}t_D/x$.*
*For $100 < t_D < 0.25\ r_{eD}^2$, $p_D \cong 0.5\ (\ln t_D + 0.80907)$.*

- For $t_D < 0.01$:

$$p_D = 2\sqrt{\frac{t_D}{\pi}} \tag{6-91}$$

- For $t_D > 100$:

$$p_D = 0.5[\ln(t_D) + 0.80907] \tag{6-92}$$

• For $0.02 < t_D < 1000$:

$$p_D = a_1 + a_2 \ln (t_D) + a_3 [\ln (t_D)]^2 + a_4 [\ln (t_D)]^3 + a_5 t_D + a_6 (t_D)^2 + a_7 (t_D)^3 + a_8/t_D \quad (6\text{-}93)$$

where

$a_1 = 0.8085064$ $a_2 = 0.29302022$ $a_3 = 3.5264177(10^{-2})$
$a_4 = -1.4036304(10^{-3})$ $a_5 = -4.7722225(10^{-4})$ $a_6 = 5.1240532(10^{-7})$
$a_7 = -2.3033017(10^{-10})$ $a_8 = -2.6723117(10^{-3})$

## Finite-Radial Reservoir

The arrival of the pressure disturbance at the well drainage boundary marks the end of the transient flow period and the beginning of the semi (pseudo)-steady state. During this flow state, the reservoir boundaries and the shape of the drainage area influence the wellbore pressure response as well as the behavior of the pressure distribution throughout the reservoir. Intuitively, one should not expect the change from the transient to the semi-steady state in this bounded (finite) system to occur instantaneously. There is a short period of time that separates the transient state from the semi-steady state that is called *late-transient state*. Due to its complexity and short duration, the late transient flow is not used in practical well test analysis.

For a finite radial system, the $p_D$-function is a function of both the dimensionless time and radius, or:

$$p_D = f\,(t_D, r_{eD})$$

where

$$r_{eD} = \frac{\text{external radius}}{\text{wellbore radius}} = \frac{r_e}{r_w} \quad (6\text{-}94)$$

Table 6-3 presents $p_D$ as a function of $t_D$ for $1.5 < r_{eD} < 10$. It should be pointed out that Van Everdingen and Hurst principally applied the $p_D$-function solution to model the performance of water influx into oil reservoirs. Thus, the authors' wellbore radius $r_w$ was in this case the external radius of the reservoir and the $r_e$ was essentially the external boundary radius of the aquifer. Therefore, the range of the $r_{eD}$ values in Table 6-3 is practical for this application.

**Table 6-3**
**$p_D$ vs. $t_D$—Finite-Radial System, Constant-Rate at the Inner Boundary**
**(*After Lee, J.,* Well Testing, *SPE Textbook Series.*)**
**(*Permission to publish by the SPE, copyright SPE, 1982*)**

| $r_{eD}$ = 1.5 | | $r_{eD}$ = 2.0 | | $r_{eD}$ = 2.5 | | $r_{eD}$ = 3.0 | | $r_{eD}$ = 3.5 | | $r_{eD}$ = 4.0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $t_D$ | $p_D$ | $t_D$ | $p_D$ | $t_D$ | $p_D$ | $t_D$ | $p_D$ | $t_D$ | $p_D$ | $t_D$ | $p_D$ |
| 0.06 | 0.251 | 0.22 | 0.443 | 0.40 | 0.565 | 0.52 | 0.627 | 1.0 | 0.802 | 1.5 | 0.927 |
| 0.08 | 0.288 | 0.24 | 0.459 | 0.42 | 0.576 | 0.54 | 0.636 | 1.1 | 0.830 | 1.6 | 0.948 |
| 0.10 | 0.322 | 0.26 | 0.476 | 0.44 | 0.587 | 0.56 | 0.645 | 1.2 | 0.857 | 1.7 | 0.968 |
| 0.12 | 0.355 | 0.28 | 0.492 | 0.46 | 0.598 | 0.60 | 0.662 | 1.3 | 0.882 | 1.8 | 0.988 |
| 0.14 | 0.387 | 0.30 | 0.507 | 0.48 | 0.608 | 0.65 | 0.683 | 1.4 | 0.906 | 1.9 | 1.007 |
| 0.16 | 0.420 | 0.32 | 0.522 | 0.50 | 0.618 | 0.70 | 0.703 | 1.5 | 0.929 | 2.0 | 1.025 |
| 0.18 | 0.452 | 0.34 | 0.536 | 0.52 | 0.628 | 0.75 | 0.721 | 1.6 | 0.951 | 2.2 | 1.059 |
| 0.20 | 0.484 | 0.36 | 0.551 | 0.54 | 0.638 | 0.80 | 0.740 | 1.7 | 0.973 | 2.4 | 1.092 |
| 0.22 | 0.516 | 0.38 | 0.565 | 0.56 | 0.647 | 0.85 | 0.758 | 1.8 | 0.994 | 2.6 | 1.123 |
| 0.24 | 0.548 | 0.40 | 0.579 | 0.58 | 0.657 | 0.90 | 0.776 | 1.9 | 1.014 | 2.8 | 1.154 |
| 0.26 | 0.580 | 0.42 | 0.593 | 0.60 | 0.666 | 0.95 | 0.791 | 2.0 | 1.034 | 3.0 | 1.184 |
| 0.28 | 0.612 | 0.44 | 0.607 | 0.65 | 0.688 | 1.0 | 0.806 | 2.25 | 1.083 | 3.5 | 1.255 |
| 0.30 | 0.644 | 0.46 | 0.621 | 0.70 | 0.710 | 1.2 | 0.865 | 2.50 | 1.130 | 4.0 | 1.324 |
| 0.35 | 0.724 | 0.48 | 0.634 | 0.75 | 0.731 | 1.4 | 0.920 | 2.75 | 1.176 | 4.5 | 1.392 |
| 0.40 | 0.804 | 0.50 | 0.648 | 0.80 | 0.752 | 1.6 | 0.973 | 3.0 | 1.221 | 5.0 | 1.460 |
| 0.45 | 0.884 | 0.60 | 0.715 | 0.85 | 0.772 | 2.0 | 1.076 | 4.0 | 1.401 | 5.5 | 1.527 |
| 0.50 | 0.964 | 0.70 | 0.782 | 0.90 | 0.792 | 3.0 | 1.328 | 5.0 | 1.579 | 6.0 | 1.594 |
| 0.55 | 1.044 | 0.80 | 0.849 | 0.95 | 0.812 | 4.0 | 1.578 | 6.0 | 1.757 | 6.5 | 1.660 |
| 0.60 | 1.124 | 0.90 | 0.915 | 1.0 | 0.832 | 5.0 | 1.828 | | | 7.0 | 1.727 |
| 0.65 | 1.204 | 1.0 | 0.982 | 2.0 | 1.215 | | | | | 8.0 | 1.861 |
| 0.70 | 1.284 | 2.0 | 1.649 | 3.0 | 1.506 | | | | | 9.0 | 1.994 |
| 0.75 | 1.364 | 3.0 | 2.316 | 4.0 | 1.977 | | | | | 10.0 | 2.127 |
| 0.80 | 1.444 | 5.0 | 3.649 | 5.0 | 2.398 | | | | | | |

| $r_{eD}$ = 4.5 | | $r_{eD}$ = 5.0 | | $r_{eD}$ = 6.0 | | $r_{eD}$ = 7.0 | | $r_{eD}$ = 8.0 | | $r_{eD}$ = 9.0 | | $r_{eD}$ = 10.0 | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t_D$ | $p_D$ | $t_D$ | $p_D$ | $t_D$ | $p_D$ | $t_D$ | $p_D$ | $t_D$ | $p_D$ | $t_D$ | $p_D$ | $t_D$ | $p_D$ |
| 2.0 | 1.023 | 3.0 | 1.167 | 4.0 | 1.275 | 6.0 | 1.436 | 8.0 | 1.556 | 10.0 | 1.651 | 12.0 | 1.732 |
| 2.1 | 1.040 | 3.1 | 1.180 | 4.5 | 1.322 | 6.5 | 1.470 | 8.5 | 1.582 | 10.5 | 1.673 | 12.5 | 1.750 |
| 2.2 | 1.056 | 3.2 | 1.192 | 5.0 | 1.364 | 7.0 | 1.501 | 9.0 | 1.607 | 11.0 | 1.693 | 13.0 | 1.768 |
| 2.3 | 1.702 | 3.3 | 1.204 | 5.5 | 1.404 | 7.5 | 1.531 | 9.5 | 1.631 | 11.5 | 1.713 | 13.5 | 1.784 |
| 2.4 | 1.087 | 3.4 | 1.215 | 6.0 | 1.441 | 8.0 | 1.559 | 10.0 | 1.653 | 12.0 | 1.732 | 14.0 | 1.801 |
| 2.5 | 1.102 | 3.5 | 1.227 | 6.5 | 1.477 | 8.5 | 1.586 | 10.5 | 1.675 | 12.5 | 1.750 | 14.5 | 1.817 |
| 2.6 | 1.116 | 3.6 | 1.238 | 7.0 | 1.511 | 9.0 | 1.613 | 11.0 | 1.697 | 13.0 | 1.768 | 15.0 | 1.832 |
| 2.7 | 1.130 | 3.7 | 1.249 | 7.5 | 1.544 | 9.5 | 1.638 | 11.5 | 1.717 | 13.5 | 1.786 | 15.5 | 1.847 |
| 2.8 | 1.144 | 3.8 | 1.259 | 8.0 | 1.576 | 10.0 | 1.663 | 12.0 | 1.737 | 14.0 | 1.803 | 16.0 | 1.862 |
| 2.9 | 1.158 | 3.9 | 1.270 | 8.5 | 1.607 | 11.0 | 1.711 | 12.5 | 1.757 | 14.5 | 1.819 | 17.0 | 1.890 |
| 3.0 | 1.171 | 4.0 | 1.281 | 9.0 | 1.638 | 12.0 | 1.757 | 13.0 | 1.776 | 15.0 | 1.835 | 18.0 | 1.917 |

(*table continued on next page*)

**Table 6-3 (*continued*)**

| $r_{eD} = 4.5$ | | $r_{eD} = 5.0$ | | $r_{eD} = 6.0$ | | $r_{eD} = 7.0$ | | $r_{eD} = 8.0$ | | $r_{eD} = 9.0$ | | $r_{eD} = 10.0$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $t_D$ | $p_D$ | $t_D$ | $p_D$ | $t_D$ | $p_D$ | $t_D$ | $p_D$ | $t_D$ | $p_D$ | $t_D$ | $p_D$ | $t_D$ | $p_D$ |
| 3.2 | 1.197 | 4.2 | 1.301 | 9.5 | 1.668 | 13.0 | 1.810 | 13.5 | 1.795 | 15.5 | 1.851 | 19.0 | 1.943 |
| 3.4 | 1.222 | 4.4 | 1.321 | 10.0 | 1.698 | 14.0 | 1.845 | 14.0 | 1.813 | 16.0 | 1.867 | 20.0 | 1.968 |
| 3.6 | 1.246 | 4.6 | 1.340 | 11.0 | 1.757 | 15.0 | 1.888 | 14.5 | 1.831 | 17.0 | 1.897 | 22.0 | 2.017 |
| 3.8 | 1.269 | 4.8 | 1.360 | 12.0 | 1.815 | 16.0 | 1.931 | 15.0 | 1.849 | 18.0 | 1.926 | 24.0 | 2.063 |
| 4.0 | 1.292 | 5.0 | 1.378 | 13.0 | 1.873 | 17.0 | 1.974 | 17.0 | 1.919 | 19.0 | 1.955 | 26.0 | 2.108 |
| 4.5 | 1.349 | 5.5 | 1.424 | 14.0 | 1.931 | 18.0 | 2.016 | 19.0 | 1.986 | 20.0 | 1.983 | 28.0 | 2.151 |
| 5.0 | 1.403 | 6.0 | 1.469 | 15.0 | 1.988 | 19.0 | 2.058 | 21.0 | 2.051 | 22.0 | 2.037 | 30.0 | 2.194 |
| 5.5 | 1.457 | 6.5 | 1.513 | 16.0 | 2.045 | 20.0 | 2.100 | 23.0 | 2.116 | 24.0 | 2.906 | 32.0 | 2.236 |
| 6.0 | 1.510 | 7.0 | 1.556 | 17.0 | 2.103 | 22.0 | 2.184 | 25.0 | 2.180 | 26.0 | 2.142 | 34.0 | 2.278 |
| 7.0 | 1.615 | 7.5 | 1.598 | 18.0 | 2.160 | 24.0 | 2.267 | 30.0 | 2.340 | 28.0 | 2.193 | 36.0 | 2.319 |
| 8.0 | 1.719 | 8.0 | 1.641 | 19.0 | 2.217 | 26.0 | 2.351 | 35.0 | 2.499 | 30.0 | 2.244 | 38.0 | 2.360 |
| 9.0 | 1.823 | 9.0 | 1.725 | 20.0 | 2.274 | 28.0 | 2.434 | 40.0 | 2.658 | 34.0 | 2.345 | 40.0 | 2.401 |
| 10.0 | 1.927 | 10.0 | 1.808 | 25.0 | 2.560 | 30.0 | 2.517 | 45.0 | 2.817 | 38.0 | 2.446 | 50.0 | 2.604 |
| 11.0 | 2.031 | 11.0 | 1.892 | 30.0 | 2.846 | | | | | 40.0 | 2.496 | 60.0 | 2.806 |
| 12.0 | 2.135 | 12.0 | 1.975 | | | | | | | 45.0 | 2.621 | 70.0 | 3.008 |
| 13.0 | 2.239 | 13.0 | 2.059 | | | | | | | 50.0 | 2.746 | 80.0 | 3.210 |
| 14.0 | 2.343 | 14.0 | 2.142 | | | | | | | 60.0 | 2.996 | 90.0 | 3.412 |
| 15.0 | 2.447 | 15.0 | 2.225 | | | | | | | 70.0 | 3.246 | 100.0 | 3.614 |

*Notes: For $t_D$ smaller than values listed in this table for a given $r_{eD}$, reservoir is infinite acting. Find $p_D$ in Table 6-2.*
*For $25 < t_D$ and $t_D$ larger than values in table.*

$$p_D \cong \frac{(\frac{1}{2} + 2t_D)}{(r_{eD}^2 - 1)} - \frac{3r_{eD}^4 - 4r_{eD}^4 \ln r_{eD} - 2r_{eD}^2 - 1}{4(r_{eD}^2 - 1)^2}$$

*For wells in rebounded reservoirs with $r_{eD}^2 \gg 1$*

$$p_D \cong \frac{2t_D}{r_{eD}^2} + \ln r_{eD} - \frac{3}{4}.$$

Chatas (1953) proposed the following mathematical expression for calculating $p_D$:

For $\mathbf{25 < t_D}$ and $\mathbf{0.25\, r_{eD}^2 < t_D}$

$$p_D = \frac{0.5 + 2t_D}{r_{eD}^2 - 1} - \frac{r_{eD}^4\,[3 - 4\ln(r_{eD})] - 2r_{eD}^2 - 1}{4\left(r_{eD}^2 - 1\right)^2} \tag{6-95}$$

A special case of Equation 6-95 arises when $r_{eD}^2 \gg 1$, then:

$$p_D = \frac{2\,t_D}{r_{eD}^2} + \ln(r_{eD}) - 0.75 \tag{6-96}$$

The computational procedure of using the $p_D$-function in determining the bottom-hole flowing pressure changing the transient flow period is summarized in the following steps:

*Step 1.* Calculate the dimensionless time $t_D$ by applying Equation 6-87.

*Step 2.* Calculate the dimensionless radius $r_{eD}$ from Equation 6-89.

*Step 3.* Using the calculated values of $t_D$ and $r_{eD}$, determine the corresponding pressure function $p_D$ from the appropriate table or equation.

*Step 4.* Solve for the pressure at the desired radius, i.e., $r_w$, by applying Equation 6-86, or:

$$p(r_w, t) = p_i - \left(\frac{Q_o B_o \mu_o}{0.00708\ k h}\right) p_D \qquad (6\text{-}97)$$

## Example 6-12

A well is producing at a constant flow rate of 300 STB/day under unsteady-state flow condition. The reservoir has the following rock and fluid properties (see Example 6-10):

| | | |
|---|---|---|
| $B_o = 1.25$ bbl/STB | $\mu_o = 1.5$ cp | $c_t = 12 \times 10^{-6}$ psi$^{-1}$ |
| $k = 60$ md | $h = 15$ ft | $p_i = 4000$ psi |
| $\phi = 15\%$ | $r_w = 0.25'$ | |

Assuming an infinite acting reservoir, i.e., $r_{eD} = \infty$, calculate the bottom-hole flowing pressure after one hour of production by using the dimensionless pressure approach.

## Solution

*Step 1.* Calculate the dimensionless time $t_D$ from Equation 6-87.

$$t_D = \frac{0.000264\ (60)\ (1)}{(0.15)(1.5)(12 \times 10^{-6})(0.25)^2} = 93{,}866.67$$

*Step 2.* Since $t_D > 100$, use Equation 6-92 to calculate the dimensionless pressure drop function:

$$p_D = 0.5\ [\ln(93{,}866.67) + 0.80907] = 6.1294$$

*Step 3.* Calculate the bottom-hole pressure after 1 hour by applying Equation 6-97:

$$p(0.25, 1) = 4000 - \left[\frac{(300)(1.25)(1.5)}{0.00708(60)(15)}\right](6.1294) = 3459\ \text{psi}$$

The above example shows that the solution as given by the $p_D$-function technique is identical to that of the $E_i$-function approach. The main difference between the two formulations is that **the $p_D$-function can be used only to calculate the pressure at radius r when the flow rate Q is constant and known.** In that case, the $p_D$-function application is essentially restricted to the wellbore radius because the rate is usually known. On the other hand, the $E_i$-function approach can be used to calculate the pressure at any radius in the reservoir by using the well flow rate Q.

It should be pointed out that, for an infinite-acting reservoir with $t_D > 100$, the $p_D$-function is related to the $E_i$-function by the following relation:

$$p_D = 0.5\left[-E_i\left(\frac{-1}{4\,t_D}\right)\right] \tag{6-98}$$

The previous example, i.e., Example 6-12, is not a practical problem, but it is essentially designed to show the physical significance of the $p_D$ solution approach. In transient flow testing, we normally record the bottom-hole flowing pressure as a function of time. Therefore, the dimensionless pressure drop technique can be used to determine one or more of the reservoir properties, e.g., k or kh, as discussed later in this chapter.

## Radial Flow of Compressible Fluids

Gas viscosity and density vary significantly with pressure and therefore the assumptions of Equation 6-76 are not satisfied for gas systems, i.e., compressible fluids. In order to develop the proper mathematical function for describing the flow of compressible fluids in the reservoir, the following two additional gas equations must be considered:

• Real density equation

$$\rho = \frac{pM}{zRT}$$

• Gas compressibility equation

$$c_g = \frac{1}{p} - \frac{1}{z}\frac{dz}{dp}$$

Combining the above two basic gas equations with that of Equation 6-68 gives:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{p}{\mu z}\frac{\partial p}{\partial r}\right) = \frac{\phi \mu c_t}{0.000264\,k}\frac{p}{\mu z}\frac{\partial p}{\partial t} \tag{6-99}$$

where $t$ = time, hr
$k$ = permeability, md
$c_t$ = total isothermal compressibility, $psi^{-1}$
$\phi$ = porosity

Al-Hussainy, Ramey, and Crawford (1966) linearize the above basic flow equation by introducing the real gas potential m(p) to Equation 6-99. Recall the previously defined m(p) equation:

$$m(p) = \int_0^p \frac{2p}{\mu z}\,dp \tag{6-100}$$

Differentiating the above relation with respect to p gives:

$$\frac{\partial m(p)}{\partial p} = \frac{2p}{\mu z} \tag{6-101}$$

Obtain the following relationships by applying the chair rule:

$$\frac{\partial m(p)}{\partial r} = \frac{\partial m(p)}{\partial p}\frac{\partial p}{\partial r} \tag{6-102}$$

$$\frac{\partial m(p)}{\partial t} = \frac{\partial m(p)}{\partial p}\frac{\partial p}{\partial t} \tag{6-103}$$

Substituting Equation 6-101 into Equations 6-102 and 6-103 gives:

$$\frac{\partial p}{\partial r} = \frac{\mu z}{2p}\frac{\partial m(p)}{\partial r} \tag{6-104}$$

and

$$\frac{\partial p}{\partial t} = \frac{\mu z}{2p}\frac{\partial m(p)}{\partial t} \tag{6-105}$$

Combining Equations 6-104 and 6-105 with 6-99 yields:

$$\frac{\partial^2 m(p)}{\partial r^2} + \frac{1}{r}\frac{\partial m(p)}{\partial r} = \frac{\phi \mu c_t}{0.000264\,k}\frac{\partial m(p)}{\partial t} \tag{6-106}$$

Equation 6-106 is the radial diffusivity equation for compressible fluids. This differential equation relates the real gas pseudopressure (real gas potential) to the time t and the radius r. Al-Hussainy, Ramey, and Crawford (1966) pointed out that in gas well testing analysis, the constant-rate solution has more practical applications than that provided by the constant-pressure solution. The authors provided the exact solution to Equation 6-106 that is commonly referred to as the m(p)-solution method. There are also two other solutions that approximate the exact solution. These two approximation methods are called the pressure-squared method and the *pressure-approximation method.* In general, there are three forms of the mathematical solution to the diffusivity equation:

- The m(p)-Solution Method (Exact Solution)
- The Pressure-Squared Method ($p^2$-Approximation Method)
- The Pressure Method (p-Approximation Method)

These three methods are presented as follows:

## The m(p)-Solution Method (Exact-Solution)

Imposing the constant-rate condition as one of the boundary conditions required to solve Equation 6-106, Al-Hussainy, et al. (1966) proposed the following exact solution to the diffusivity equation:

$$m(p_{wf}) = m(p_i) - 57{,}895.3\left(\frac{p_{sc}}{T_{sc}}\right)\left(\frac{Q_g T}{kh}\right) \left[\log\left(\frac{kt}{\phi \mu_i c_{ti} r_w^2}\right) - 3.23\right] \quad (6\text{-}107)$$

where $p_{wf}$ = bottom-hole flowing pressure, psi
$p_e$ = initial reservoir pressure
$Q_g$ = gas flow rate, Mscf/day
$t$ = time, hr
$k$ = permeability, md
$p_{sc}$ = standard pressure, psi
$T_{sc}$ = standard temperature, °R
$T$ = reservoir temperature
$r_w$ = wellbore radius, ft
$h$ = thickness, ft
$\mu_i$ = gas viscosity at the initial pressure, cp
$c_{ti}$ = total compressibility coefficient at $p_i$, $psi^{-1}$
$\phi$ = porosity

When $p_{sc}$ = 14.7 psia and $T_{sc}$ = 520°R, Equation 6-107 reduces to:

$$m(p_{wf}) = m(p_i) - \left(\frac{1637 Q_g T}{kh}\right)\left[\log\left(\frac{kt}{\phi \mu_i c_{ti} r_w^2}\right) - 3.23\right] \quad (6\text{-}108)$$

Equation 6-108 can be written equivalently in terms of the dimensionless time $t_D$ as:

$$m(p_{wf}) = m(p_i) - \left(\frac{1637 Q_g T}{kh}\right)\left[\log\left(\frac{4t_D}{\gamma}\right)\right] \quad (6\text{-}109)$$

The dimensionless time is defined previously by Equation 6-86 as:

$$t_D = \frac{0.000264\,kt}{\phi\,\mu_i\,c_{ti}\,r_w^2}$$

The parameter $\gamma$ is called Euler's constant and given by:

$$\gamma = e^{0.5772} = 1.781 \tag{6-110}$$

The solution to the diffusivity equation as given by Equations 6-108 and 6-109 expresses the bottom-hole real gas pseudopressure as a function of the transient flow time t. The solution as expressed in terms of m(p) is recommended mathematical expression for performing gas-well pressure analysis due to its applicability in all pressure ranges.

The radial gas diffusivity equation can be expressed in a dimensionless form in terms of the dimensionless real gas pseudopressure drop $\psi_D$. The solution to the dimensionless equation is given by:

$$m(p_{wf}) = m(p_i) - \left(\frac{1422\,Q_g\,T}{kh}\right)\psi_D \tag{6-111}$$

where $Q_g$ = gas flow rate, Mscf/day
k = permeability, md

The dimensionless pseudopressure drop $\psi_D$ can be determined as a function of $t_D$ by using the appropriate expression of Equations 6-91 through 6-96. When $t_D$ > 100, the $\psi_D$ can be calculated by applying Equation 6-82, or:

$$\psi_D = 0.5\,[\ln(t_D) + 0.80907] \tag{6-112}$$

## Example 6-13

A gas well with a wellbore radius of 0.3 ft is producing at a constant flow rate of 2,000 Mscf/day under transient flow conditions. The initial reservoir pressure (shut-in pressure) is 4,400 psi at 140°F. The formation permeability and thickness are 65 md and 15 ft, respectively. The porosity is recorded as 15%. Example 6-7 documents the properties of the gas as well as values of m(p) as a function of pressures. The table is reproduced below for convenience:

| p | $\mu_g$ (cp) | z | m(p), $psi^2/cp$ |
|---|---|---|---|
| 0 | 0.01270 | 1.000 | 0.000 |
| 400 | 0.01286 | 0.937 | $13.2 \times 10^6$ |
| 800 | 0.01390 | 0.882 | $52.0 \times 10^6$ |
| 1200 | 0.01530 | 0.832 | $113.1 \times 10^6$ |
| 1600 | 0.01680 | 0.794 | $198.0 \times 10^6$ |
| 2000 | 0.01840 | 0.770 | $304.0 \times 10^6$ |
| 2400 | 0.02010 | 0.763 | $422.0 \times 10^6$ |
| 2800 | 0.02170 | 0.775 | $542.4 \times 10^6$ |
| 3200 | 0.02340 | 0.797 | $678.0 \times 10^6$ |
| 3600 | 0.02500 | 0.827 | $816.0 \times 10^6$ |
| 4000 | 0.02660 | 0.860 | $950.0 \times 10^6$ |
| 4400 | 0.02831 | 0.896 | $1089.0 \times 10^6$ |

Assuming that the initial total isothermal compressibility is $3 \times 10^{-4}$ $psi^{-1}$, calculate the bottom-hole flowing pressure after 1.5 hours.

*Step 1.* Calculate the dimensionless time $t_D$

$$t_D = \frac{(0.000264)\,(65)\,(1.5)}{(0.15)\,(0.02831)\,(3 \times 10^{-4})\,(0.3^2)} = 224{,}498.6$$

*Step 2.* Solve for $m(p_{wf})$ by using Equation 6-109

$$m(p_{wf}) = 1089 \times 10^6 - \frac{(1637)\,(2000)\,(600)}{(65)\,(15)}\left[\log\left(\frac{(4)\,224{,}498.6}{e^{0.5772}}\right)\right]$$
$$= 1077.5\ (10^6)$$

*Step 3.* From the given PVT data, interpolate using the value of $m(p_{wf})$ to give a corresponding $p_{wf}$ of 4,367 psi.

An identical solution can be obtained by applying the $\psi_D$ approach as shown below:

*Step 1.* Calculate $\psi_D$ from Equation 6-112

$$\psi_D = 0.5\ [\ln\,(224{,}498.6) + 0.8090] = 6.565$$

*Step 2.* Calculate $m(p_{wf})$ by using Equation 6-111

$$m(p_{wf}) = 1089 \times 10^6 - \left(\frac{1422(2000)(600)}{(65)(15)}\right)(6.565) = 1077.5 \times 10^6$$

## The Pressure-Squared Approximation Method ($p^2$-method)

The first approximation to the exact solution is to remove the pressure-dependent term ($\mu z$) outside the integral that defines $m(p_{wf})$ and $m(p_i)$ to give:

$$m(p_i) - m(p_{wf}) = \frac{2}{\bar{\mu}\,\bar{z}} \int_{p_{wf}}^{p_i} p\,dp \tag{6-113}$$

or

$$m(p_i) - m(p_{wf}) = \frac{p_i^2 - p_{wf}^2}{\bar{\mu}\,\bar{z}} \tag{6-114}$$

The bars over $\mu$ and z represent the values of the gas viscosity and deviation factor as evaluated at the average pressure $\bar{p}$. This average pressure is given by:

$$\bar{p} = \sqrt{\frac{p_i^2 + p_{wf}^2}{2}} \tag{6-115}$$

Combining Equation 6-114 with Equation 6-108, 6-109, or 6-111 gives:

$$p_{wf}^2 = p_i^2 - \left(\frac{1637 Q_g T \bar{\mu}\,\bar{z}}{kh}\right)\left[\log\left(\frac{kt}{\phi \mu_i c_{ti} r_w^2}\right) - 3.23\right] \tag{6-116}$$

or

$$p_{wf}^2 = p_i^2 - \left(\frac{1637 Q_g T \bar{\mu}\,\bar{z}}{kh}\right)\left[\log\left(\frac{4 t_D}{\gamma}\right)\right] \tag{6-117}$$

or, equivalently:

$$p_{wf}^2 = p_i^2 - \left(\frac{1422\,Q_g\,T\,\bar{\mu}\,\bar{z}}{kh}\right)\psi_D \tag{6-118}$$

The above approximation solution forms indicate that the product (μz) is assumed constant at the average pressure $\bar{p}$. This effectively limits the applicability of the $p^2$-method to reservoir pressures < 2000. It should be pointed out that when the $p^2$-method is used to determine $p_{wf}$ it is perhaps sufficient to set $\bar{\mu}\,\bar{z} = \mu_i\,z$.

## Example 6-14

A gas well is producing at a constant rate of 7,454.2 Mscf/day under transient flow conditions. The following data are available:

$k = 50$ md  $h = 10$ ft  $\phi = 20\%$  $p_i = 1600$ psi
$T = 600°R$  $r_w = 0.3$ ft  $c_{ti} = 6.25 \times 10^{-4}$ $psi^{-1}$

The gas properties are tabulated below:

| p | $\mu_g$, cp | z | m(p), $psi^2/cp$ |
|---|---|---|---|
| 0 | 0.01270 | 1.000 | 0.000 |
| 400 | 0.01286 | 0.937 | $13.2 \times 10^6$ |
| 800 | 0.01390 | 0.882 | $52.0 \times 10^6$ |
| 1200 | 0.01530 | 0.832 | $113.1 \times 10^6$ |
| 1600 | 0.01680 | 0.794 | $198.0 \times 10^6$ |

Calculate the bottom-hole flowing pressure after 4 hours by using:

a. The m(p)-method
b. The $p^2$-method

## Solution

### *a. The m(p)-method*

*Step 1.* Calculate $t_D$

$$t_D = \frac{0.000264(50)(4)}{(0.2)(0.0168)(6.25\times10^{-4})(0.3^2)} = 279{,}365.1$$

*Step 2.* Calculate $\psi_D$:

$$\psi_D = 0.5\ [\text{Ln}(279365.1) + 0.80907] = 6.6746$$

*Step 3.* Solve for $m(p_{wf})$ by applying Equation 6-111:

$$m(p_{wf}) = (198 \times 10^6) - \left[\frac{1422\,(7454.2)\,(600)}{(50)\,(10)}\right] 6.6746 = 113.1 \times 10^6$$

The corresponding value of $p_{wf} = 1200$ psi.

### *b. The $p^2$-method*

*Step 1.* Calculate $\psi_D$ by applying Equation 6-112:

$$\psi_D = 0.5\,[\ln(279365.1) + 0.80907] = 6.6477$$

*Step 2.* Calculate $p_{wf}^2$ by applying Equation 6-118:

$$p_{wf}^2 = 1600^2 - \left[\frac{(1422)\,(7454.2)\,(600)\,(0.0168)\,(0.794)}{(50)\,(10)}\right] \times 6.6747 = 1{,}427{,}491$$

$$p_{wf} = 1195 \text{ psi}$$

*Step 3.* The absolute average error is 0.4%

## The Pressure-Approximation Method

The second method of approximation to the exact solution of the radial flow of gases is to treat the gas as a *pseudoliquid.*

Recalling the gas formation volume factor $B_g$ as expressed in bbl/scf is given by:

$$B_g = \left(\frac{p_{sc}}{5.615 T_{sc}}\right)\left(\frac{zT}{p}\right)$$

Solving the above expression for p/z gives:

$$\frac{p}{z} = \left(\frac{T p_{sc}}{5.615\, T_{sc}}\right)\left(\frac{1}{B_g}\right)$$

The difference in the real gas pseudopressure is given by:

$$m(p_i)-(p_{wf})=\int_{p_{wf}}^{p_i}\frac{2p}{\mu z}\,dp$$

Combining the above two expressions gives:

$$m(p_i)-m(p_{wf})=\frac{2T\,p_{sc}}{5.615T_{sc}}\int_{p_{wf}}^{p_i}\left(\frac{1}{\mu B_g}\right)dp \qquad (6\text{-}119)$$

Fetkovich (1973) suggested that at high pressures (p > 3,000), $1/\mu B_g$ is nearly constant as shown schematically in Figure 6-22. Imposing Fetkovich's condition on Equation 6-119 and integrating gives:

$$m(p_i)-m(p_{wf})=\frac{2T\,p_{sc}}{5.615T_{sc}\,\bar{\mu}\,\bar{B}_g}(p_i-p_{wf}) \qquad (6\text{-}120)$$

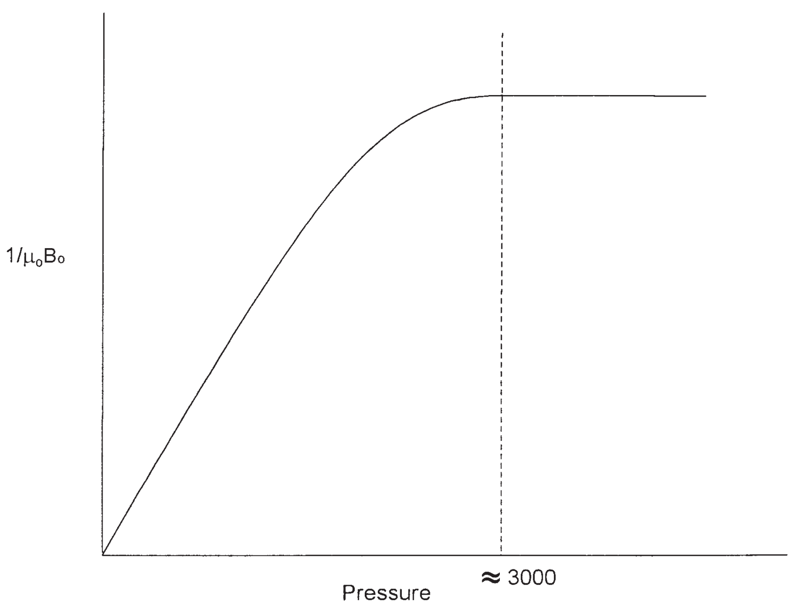



**Figure 6-22.** $1/\mu_o B_o$ vs. pressure.

Combining Equation 6-120 with Equation 6-108, 6-109, or 6-111 gives:

$$p_{wf} = p_i - \left(\frac{162.5 \times 10^3\, Q_g\, \bar{\mu}\, \bar{B}_g}{kh}\right)\left[\log\left(\frac{kt}{\phi \bar{\mu}\, \bar{c}_t\, r_w^2}\right) - 3.23\right] \tag{6-121}$$

or

$$p_{wf} = p_i - \left(\frac{162.5\,(10^3)\, Q_g\, \bar{\mu}\, \bar{B}_g}{kh}\right)\left[\log\left(\frac{4\,t_D}{\gamma}\right)\right] \tag{6-122}$$

or equivalently in terms of dimensionless pressure drop:

$$p_{wf} = p_i - \left(\frac{141.2\,(10^3)\, Q_g\, \bar{\mu}\, \bar{B}_g}{kh}\right) p_D \tag{6-123}$$

where $Q_g$ = gas flow rate, Mscf/day
$k$ = permeability, md
$\bar{B}_g$ = gas formation volume factor, bbl/scf
$t$ = time, hr
$p_D$ = dimensionless pressure drop
$t_D$ = dimensionless time

It should be noted that the gas properties, i.e., $\mu$, $B_g$, and $c_t$, are evaluated at pressure $\bar{p}$ as defined below:

$$\bar{p} = \frac{p_i + p_{wf}}{2} \tag{6-124}$$

Again, this method is only limited to applications above 3,000 psi. When solving for $p_{wf}$, it might be sufficient to evaluate the gas properties at $p_i$.

## Example 6-15

Resolve Example 6-13 by using the p-approximation method and compare with the exact solution.

**Solution**

*Step 1.* Calculate the dimensionless time $t_D$.

$$t_D = \frac{(0.000264)(65)(1.5)}{(0.15)(0.02831)(3 \times 10^{-4})(0.3^2)} = 224,498.6$$

*Step 2.* Calculate $B_g$ at $p_i$.

$$B_g = 0.00504 \frac{(0.896)(600)}{4400} = 0.0006158 \text{ bbl/scf}$$

*Step 3.* Calculate the dimensionless pressure $p_D$ by applying Equation 6-92.

$$p_D = 0.5[\ln(224,498.6) + 0.80907] = 6.565$$

*Step 4.* Approximate $p_{wf}$ from Equation 6-123.

$$p_{wf} = 4400 - \left[\frac{141.2 \times 10^3 (2000)(0.02831)(0.0006158)}{(65)(15)}\right] 6.565$$
$$= 4367 \text{ psi}$$

The solution is identical to that of the exact solution.

It should be pointed that Examples 6-10 through 6-15 are designed to illustrate the use of different solution methods. These examples are not practical, however, because in transient flow analysis, the bottom-hole flowing pressure is usually available as a function of time. All the previous methodologies are essentially used to characterize the reservoir by determining the permeability k or the permeability-thickness product (kh).

## PSEUDOSTEADY-STATE FLOW

In the unsteady-state flow cases discussed previously, it was assumed that a well is located in a very large reservoir and producing at a constant flow rate. This rate creates a pressure disturbance in the reservoir that travels throughout this infinite-size reservoir. During this transient flow period, reservoir boundaries have no effect on the pressure behavior of the well. Obviously, the time period where this assumption can be imposed is often very short in length. As soon as the pressure disturbance reaches all drainage boundaries, it ends the transient (unsteady-state) flow regime. A different flow regime begins that is called **pseudosteady (semisteady)-state flow.** It is necessary at this point to impose different boundary conditions on the diffusivity equation and derive an appropriate solution to this flow regime.

Consider Figure 6-23, which shows a well in radial system that is producing at a constant rate for a long enough period that eventually affects the entire drainage area. During this semisteady-state flow, the change in pressure with time becomes the same throughout the drainage area. Section B in Figure 6-23 shows that the pressure distributions become paralleled at successive time periods. Mathematically, this important condition can be expressed as:

$$\left(\frac{\partial p}{\partial t}\right)_r = \text{constant}$$

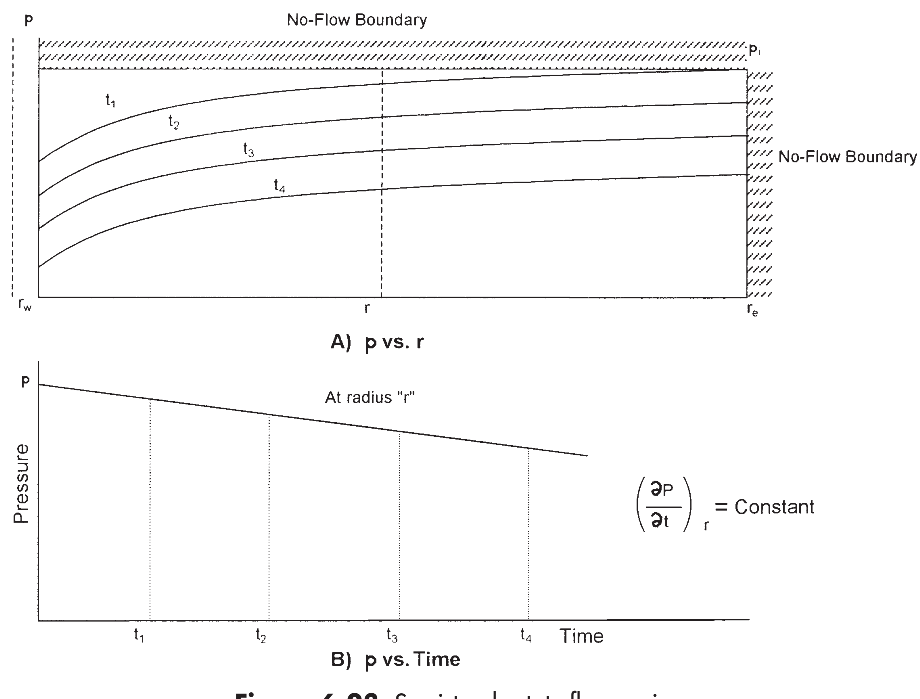



**Figure 6-23.** Semisteady-state flow regime.

The constant referred to in the above equation can be obtained from a simple material balance using the definition of the compressibility, thus:

$$c = \frac{-1}{V}\frac{dV}{dp}$$

Arranging:

$$cVdp = -dV$$

Differentiating with respect to time t:

$$cV\frac{dp}{dt} = -\frac{dV}{dt} = q$$

or

$$\frac{dp}{dt} = -\frac{q}{cV}$$

Expressing the pressure decline rate dp/dt in the above relation in psi/hr gives:

$$\frac{dp}{dt} = -\frac{q}{24\,cV} = -\frac{Q_o\,B_o}{24\,cV} \tag{6-125}$$

where q = flow rate, bbl/day
$Q_o$ = flow rate, STB/day
dp/dt = pressure decline rate, psi/hr
V = pore volume, bbl

For a radial drainage system, the pore volume is given by:

$$V = \frac{\pi\, r_e^2\, h\phi}{5.615} = \frac{Ah\phi}{5.615} \tag{6-126}$$

where A = drainage area, $ft^2$

Combining Equation 6-127 with Equation 6-126 gives:

$$\frac{dp}{dt} = -\frac{0.23396\,q}{c_t\,\pi\,r_e^2\,h\phi} = \frac{-0.23396\,q}{c_t\,Ah\phi} \tag{6-127}$$

Examination of the above expression reveals the following important characteristics of the behavior of the pressure decline rate dp/dt during the semisteady-state flow:

- The reservoir pressure declines at a higher rate with an increase in the fluids production rate
- The reservoir pressure declines at a slower rate for reservoirs with higher total compressibility coefficients
- The reservoir pressure declines at a lower rate for reservoirs with larger pore volumes

## Example 6-16

An oil well is producing at a constant oil flow rate of 1,200 STB/day under a semisteady-state flow regime. Well testing data indicate that the pressure is declining at a constant rate of 4.655 psi/hr. The following additional data are available:

$h = 25$ ft $\quad\quad \phi = 15\% \quad\quad B_o = 1.3$ bbl/STB
$c_t = 12 \times 10^{-6}\ psi^{-1}$

Calculate the well drainage area.

## Solution

- $q = Q_o\,B_o$
- $q = (1200)\,(1.3) = 1560$ bb/day
- Apply Equation 6-128 to solve for A.

$$-4.655 = -\frac{0.23396\,(1560)}{(12 \times 10^{-6})(A)(25)(0.15)}$$

$A = 1{,}742{,}400\ ft^2$

or

$$A = 1,742,400 / 43,560 = 40 \text{ acres}$$

Matthews, Brons, and Hazebroek (1954) pointed out that once the reservoir is producing under the *semisteady-state condition,* each well will drain from within its own no-flow boundary independently of the other wells. For this condition to prevail, the pressure decline rate dp/dt must be approximately constant throughout the entire reservoir, otherwise flow would occur across the boundaries causing a readjustment in their positions. Because the pressure at every point in the reservoir is changing at the same rate, it leads to the conclusion that the average reservoir pressure is changing at the same rate. This average reservoir pressure is essentially set equal to the volumetric average reservoir pressure $\bar{p}_r$. It is the pressure that is used to perform flow calculations during the semisteady state flowing condition. In the above discussion, $\bar{p}_r$ indicates that, in principal, Equation 6-128 can be used to estimate by replacing the pressure decline rate dp/dt with $(p_i - \bar{p}_r)/t$, or:

$$p_i - \bar{p}_r = \frac{0.23396qt}{c_t Ah\phi}$$

or

$$\bar{p}_r = p_i - \frac{0.23396 q t}{c_t Ah\phi} \tag{6-128}$$

where t is approximately the elapsed time since the end of the transient flow regime to the time of interest.

It should be noted that when performing material balance calculations, the volumetric average pressure of the entire reservoir is used to calculate the fluid properties. This pressure can be determined from the individual well drainage properties as follows:

$$\bar{p}_r = \frac{\sum_i \bar{p}_{ri} V_i}{\sum_i V_i} \tag{6-129}$$

in which $V_i$ = pore volume of the ith drainage volume
$\bar{p}_{ri}$ = volumetric average pressure within the ith drainage volume.

Figure 6-24 illustrates the concept of the volumetric average pressure. In practice, the $V_i$'s are difficult to determine and, therefore, it is common to use the flow rate $q_i$ in Equation 6-129.

$$\bar{p}_r = \frac{\sum_i (\bar{p}_{ri}\, q_i)}{\sum_i q_i} \tag{6-130}$$

The flow rates are measured on a routing basis throughout the lifetime of the field, thus facilitating the calculation of the volumetric average reservoir pressure, $\bar{p}_r$. Alternatively, the average reservoir pressure can be expressed in terms of the individual well's average drainage pressure decline rates and fluid flow rates by:

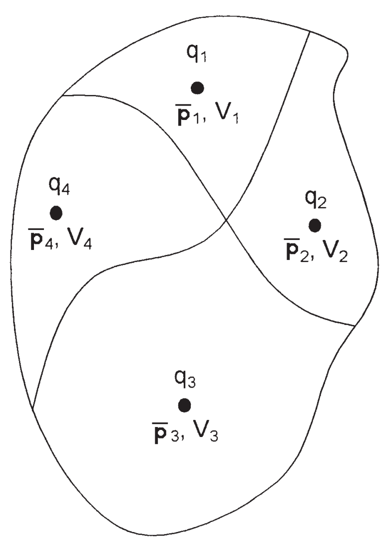



**Figure 6-24.** Volumetric average reservoir pressure.

$$\bar{p}_r = \frac{\sum_j [(\bar{p}q)_j / (\partial \bar{p}/\partial t)_j]}{\sum_j [q_j / (\partial \bar{p}/\partial t)_j]}$$

However, since the material balance equation is usually applied at regular intervals of 3 to 6 months (i.e., Δt = 3–6 months), throughout the life of the field, the average field pressure can be expressed in terms of the incremental net change in underground fluid withdrawal, Δ(F), as:

$$\bar{p}_r = \frac{\sum_j \frac{\bar{p}_j \Delta(F)_j}{\Delta \bar{p}_j}}{\sum_j \frac{\Delta(F)_j}{\Delta \bar{p}_j}}$$

Where the total underground fluid withdrawals at time t and t + Δt are given by:

$$F_t = \int_0^t [Q_o B_o + Q_w B_w + (Q_g - Q_o R_s - Q_w R_{sw}) B_g] dt$$

$$F_{t+\Delta t} = \int_0^{t+\Delta t} [Q_o B_o + Q_w B_w + (Q_g - Q_o R_s - Q_w R_{sw}) B_g] dt$$

with

$$\Delta(F) = F_{t+\Delta t} - F_t$$

where $R_s$ = gas solubility, scf/STB
$R_{sw}$ = gas solubility in the water, scf/STB
$B_g$ = gas formation volume factor, bbl/scf
$Q_o$ = oil flow rate, STB/day
$q_o$ = oil flow rate, bbl/day
$Q_w$ = water flow rate, STB/day
$q_w$ = water flow rate, bbl/day
$Q_g$ = gas flow rate, scf/day

The practical applications of using the pseudosteady-state flow condition to describe the flow behavior of the following two types of fluids are presented below:

- Radial flow of slightly compressible fluids
- Radial flow of compressible fluids

## Radial Flow of Slightly Compressible Fluids

The diffusivity equation as expressed by Equation 6-73 for the transient flow regime is:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = \left(\frac{\phi \mu c_t}{0.000264k}\right)\frac{\partial p}{\partial t}$$

For the semisteady-state flow, the term $(\partial p/\partial t)$ is constant and is expressed by Equation 6-128. Substituting Equation 6-128 into the diffusivity equation gives:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = \left(\frac{\phi \mu c_t}{0.000264k}\right)\left(\frac{-0.23396q}{c_t\,Ah\,\phi}\right)$$

or

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} = \frac{-887.22\,q\mu}{Ahk} \tag{6-131}$$

Equation 6-132 can be expressed as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) = -\frac{887.22\,q\mu}{(\pi r_e^2)hk}$$

Integrating the above equation gives:

$$r\frac{\partial p}{\partial r} = -\frac{887.22\,q\mu}{\left(\pi r_e^2\right)hk}\left(\frac{r^2}{2}\right) + c_1$$

where $c_1$ is the constant of the integration and can be evaluated by imposing the outer no-flow boundary condition [i.e., $(\partial p/\partial r)_{r_e} = 0$] on the above relation to give:

$$c_1 = \frac{141.2\,q\mu}{\pi hk}$$

Combining the above two expressions gives:

$$\frac{\partial p}{\partial r} = \frac{141.2\, q\mu}{hk}\left(\frac{1}{r} - \frac{r}{r_e^2}\right)$$

Integrating again:

$$\int_{p_{wf}}^{p_i} dp = \frac{141.2\, q\mu}{hk} \int_{rw}^{re}\left(\frac{1}{r} - \frac{r}{r_e^2}\right) dr$$

Performing the above integration and assuming $(r_w^2/r_e^2)$ is negligible gives:

$$(p_i - p_{wf}) = \frac{141.2\, q\mu}{kh}\left[\ln\left(\frac{r_e}{r_w}\right) - \frac{1}{2}\right] \tag{6-132}$$

A more appropriate form of the above is to solve for the flow rate, to give:

$$Q = \frac{0.00708\, kh\, (p_i - p_{wf})}{\mu B\left[\ln\left(\frac{r_e}{r_w}\right) - 0.5\right]} \tag{6-133}$$

where Q = flow rate, STB/day
B = formation volume factor, bbl/STB
k = permeability, md

The volumetric average reservoir pressure $\bar{p}_r$ is commonly used in calculating the liquid flow rate under the semisteady-state flowing condition. Introducing the $\bar{p}_r$ into Equation 6-134 gives:

$$Q = \frac{0.00708\, kh\, (\bar{p}_r - p_{wf})}{\mu B\left[\ln\left(\frac{r_e}{r_w}\right) - 0.75\right]} \tag{6-134}$$

Note that:

$$\ln\left(\frac{0.471 r_e}{r_w}\right) = \ln\left(\frac{r_e}{r_w}\right) - 0.75$$

The above observation suggests that the volumetric average pressure $\bar{p}_r$ occurs at about 47% of the drainage radius during the semisteady-state condition.

It is interesting to notice that the dimensionless pressure $p_D$ solution to the diffusivity equation can be used to derive Equation 6-135. The $p_D$ function for a bounded reservoir was given previously by Equation 6-96 for a bounded system as:

$$p_D = \frac{2t_D}{r_{eD}^2} + \ln(r_{eD}) - 0.75$$

where the above three dimensionless parameters are given by Equations 6-86 through 6-88 as:

$$p_D = \frac{\dfrac{(p_i - p_{wf})}{Q B\mu}}{0.00708\, k\, h}$$

$$t_D = \frac{0.000264\, k\, t}{\phi\, \mu\, c_t\, r_w^2}$$

$$r_{eD} = \frac{r_e}{r_w}$$

Combining the above four relationships gives:

$$p_{wf} = p_i - \frac{Q B\mu}{0.00708\, k\, h}\left[\frac{0.0005274\, k\, t}{\phi\, \mu\, c_t\, r_e^2} + \ln\left(\frac{r_e}{r_w}\right) - 0.75\right]$$

Solving Equation 6-129 for the time t gives:

$$t = \frac{c_t\ A\ h\ \phi\ (p_i - \bar{p}_r)}{0.23396\ Q\ B} = \frac{c_t\ (\pi\ r_e^2)\ h\ \phi\ (p_i - \bar{p}_r)}{0.23396\ Q\ B}$$

Combining the above two equations and solving for the flow rate Q yields:

$$Q = \frac{0.00708\ k\ h\ (\bar{p}_r - p_{wf})}{\mu B \left[ \ln\left(\frac{r_e}{r_w}\right) - 0.75 \right]}$$

It should be pointed out that the pseudosteady-state flow occurs regardless of the geometry of the reservoir. Irregular geometries also reach this state when they have been produced long enough for the entire drainage area to be affected.

Rather than developing a separate equation for each geometry, Ramey and Cobb (1971) introduced a correction factor that is called the **shape factor,** $C_A$, which is designed to account for the deviation of the drainage area from the ideal circular form. The shape factor, as listed in Table 6-4, accounts also for the location of the well within the drainage area. Introducing $C_A$ into Equation 6-132 and performing the solution procedure gives the following two solutions:

- In terms of the volumetric average pressure $\bar{p}_r$:

$$p_{wf} = \bar{p}_r - \frac{162.6 QB\mu}{kh} \log\left[\frac{4A}{1.781 C_A r_w^2}\right] \tag{6-135}$$

- In terms of the initial reservoir pressure $p_i$:

The changes in the average reservoir pressure as a function of time and initial reservoir pressure pi is given by Equation 6-129; as:

$$\bar{p}_r = p_i - \frac{0.23396\, q\, t}{c_t A h \phi}$$

Combining the above equation with Equation 6-136 gives:

$$p_{wf} = \left[p_i - \frac{0.23396\, Q\, Bt}{Ah\phi c_t}\right] - \frac{162.6\, Q\, B\mu}{kh} \log\left[\frac{4A}{1.781 C_A r_w^2}\right] \quad (6\text{-}136)$$

where k = permeability, md
A = drainage area, $ft^2$
$C_A$ = shape factor
Q = flow rate, STB/day
t = time, hr
$c_t$ = total compressibility coefficient, $psi^{-1}$

Equation 6-136 can be arranged to solve for Q to give:

$$Q = \frac{kh\,(\bar{p}_r - p_{wf})}{162.6\, B\mu\, \log\left[\frac{4A}{1.781 C_A r_w^2}\right]} \quad (6\text{-}137)$$

It should be noted that if Equation 6-138 is applied to a circular reservoir of a radius $r_e$, then:

$$A = \pi r_e^2$$

and the shape factor for a circular drainage area as given in Table 6-3 is:

$$C_A = 31.62$$

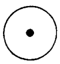
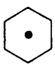
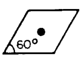
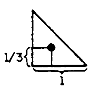
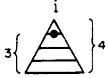
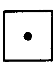
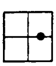
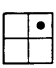
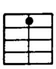
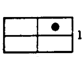
Substituting in Equation 6-138, it reduces to:

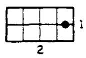
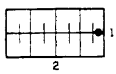
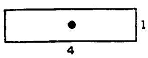
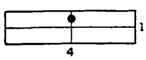
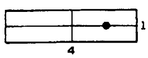
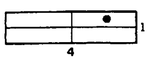
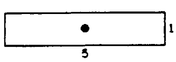
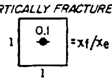
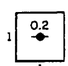
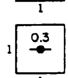
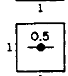
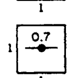
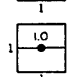
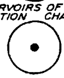
$$p_{wf} = \bar{p}_r - \left(\frac{QB\mu}{0.00708\, kh}\right)\left[\ln\left(\frac{r_e}{r_w}\right) - 0.75\right]$$

The above equation is identical to that of Equation 6-135.

(*text continued on page 427*)

**Table 6-4**
**Shape Factors for Various Single-Well Drainage Areas**
**(*After Earlougher, R.,* Advances in Well Test Analysis,**
***permission to publish by the SPE, copyright SPE, 1977*)**

| In Bounded Reservoirs | $C_A$ | $\ln C_A$ | $\frac{1}{2}\ln\left(\frac{2.2458}{C_A}\right)$ | Exact for $t_{DA}$ > | Less Than 1% Error For $t_{DA}$ > | Use Infinite System Solution with Less Than 1% Error for $t_{DA}$ < |
|---|---|---|---|---|---|---|
|  | 31.62 | 3.4538 | −1.3224 | 0.1 | 0.06 | 0.10 |
|  | 31.6 | 3.4532 | −1.3220 | 0.1 | 0.06 | 0.10 |
|  | 27.6 | 3.3178 | −1.2544 | 0.2 | 0.07 | 0.09 |
|  | 27.1 | 3.2995 | −1.2452 | 0.2 | 0.07 | 0.09 |
|  | 21.9 | 3.0865 | −1.1387 | 0.4 | 0.12 | 0.08 |
|  | 0.098 | −2.3227 | +1.5659 | 0.9 | 0.60 | 0.015 |
|  | 30.8828 | 3.4302 | −1.3106 | 0.1 | 0.05 | 0.09 |
|  | 12.9851 | 2.5638 | −0.8774 | 0.7 | 0.25 | 0.03 |
|  | 4.5132 | 1.5070 | −0.3490 | 0.6 | 0.30 | 0.025 |
|  | 3.3351 | 1.2045 | −0.1977 | 0.7 | 0.25 | 0.01 |
|  | 21.8369 | 3.0836 | −1.1373 | 0.3 | 0.15 | 0.025 |
|  | 10.8374 | 2.3830 | −0.7870 | 0.4 | 0.15 | 0.025 |
|  | 4.5141 | 1.5072 | −0.3491 | 1.5 | 0.50 | 0.06 |
|  | 2.0769 | 0.7309 | −0.0391 | 1.7 | 0.50 | 0.02 |
|  | 3.1573 | 1.1497 | −0.1703 | 0.4 | 0.15 | 0.005 |

| In Bounded Reservoirs | $C_A$ | $\ln C_A$ | $\frac{1}{2}\ln\left(\frac{2.2458}{C_A}\right)$ | Exact for $t_{DA} >$ | Less Than 1% Error For $t_{DA} >$ | Use Infinite System Solution with Less Than 1% Error for $t_{DA} <$ |
|---|---|---|---|---|---|---|
|  | 0.5813 | −0.5425 | +0.6758 | 2.0 | 0.60 | 0.02 |
|  | 0.1109 | −2.1991 | +1.5041 | 3.0 | 0.60 | 0.005 |
|  | 5.3790 | 1.6825 | −0.4367 | 0.8 | 0.30 | 0.01 |
|  | 2.6896 | 0.9894 | −0.0902 | 0.8 | 0.30 | 0.01 |
|  | 0.2318 | −1.4619 | +1.1355 | 4.0 | 2.00 | 0.03 |
|  | 0.1155 | −2.1585 | +1.4838 | 4.0 | 2.00 | 0.01 |
|  | 2.3606 | 0.8589 | −0.0249 | 1.0 | 0.40 | 0.025 |
| IN VERTICALLY FRACTURED RESERVOIRS | | Use $(x_e/x_f)^2$ in place of $A/r_w^2$ for fractured systems | | | | |
|  | 2.6541 | 0.9761 | −0.0835 | 0.175 | 0.08 | cannot use |
|  | 2.0348 | 0.7104 | +0.0493 | 0.175 | 0.09 | cannot use |
|  | 1.9986 | 0.6924 | +0.0583 | 0.175 | 0.09 | cannot use |
|  | 1.6620 | 0.5080 | +0.1505 | 0.175 | 0.09 | cannot use |
|  | 1.3127 | 0.2721 | +0.2685 | 0.175 | 0.09 | cannot use |
|  | 0.7887 | −0.2374 | +0.5232 | 0.175 | 0.09 | cannot use |
| IN WATER-DRIVE RESERVOIRS | | | | | | |
|  | 19.1 | 2.95 | −1.07 | — | — | — |
| IN RESERVOIRS OF UNKNOWN PRODUCTION CHARACTER | | | | | | |
|  | 25.0 | 3.22 | −1.20 | — | — | — |

(*text continued from page 424*)

## Example 6-17

An oil well is developed on the center of a 40-acre square drilling pattern. The well is producing at a constant flow rate of 800 STB/day under a semisteady-state condition. The reservoir has the following properties:

| | | |
|---|---|---|
| $\phi = 15\%$ | $h = 30$ ft | $k = 200$ md |
| $\mu = 1.5$ cp | $B_o = 1.2$ bbl/STB | $c_t = 25 \times 10^{-6}$ psi-1 |
| $p_i = 4500$ psi | $r_w = 0.25$ ft | $A = 40$ acres |

a. Calculate and plot the bottom-hole flowing pressure as a function of time.
b. Based on the plot, calculate the pressure decline rate. What is the decline in the average reservoir pressure from t = 10 to t = 200 hr?

## Solution

a. $p_{wf}$ calculations:

*Step 1.* From Table 6-3, determine $C_A$:

$$C_A = 30.8828$$

*Step 2.* Convert the area A from acres to $ft^2$:

$$A = (40)\ (43{,}560) = 1{,}742{,}400\ ft^2$$

*Step 3.* Apply Equation 6-137:

$$P_{wf} = 4500 - 1.719\ t - 58.536\ \log\ (2{,}027{,}436)$$

or

$$p_{wf} = 4493.69 - 1.719\ t$$

*Step 4.* Calculate $p_{wf}$ at different assumed times.

| t, hr | $P_{wf}$ = 44369 − 1.719 t |
|---|---|
| 10 | 4476.50 |
| 20 | 4459.31 |
| 50 | 4407.74 |
| 100 | 4321.79 |
| 200 | 4149.89 |

*Step 5.* Present the results of Step 4 in a graphical form as shown in Figure 6-25.

b. It is obvious from Figure 6-25 and the above calculation that the bottom-hole flowing pressure is declining at a rate of 1.719 psi/hr, or:

$$\frac{dp}{dt} = -1.719 \text{ psi/hr}$$

The significance of this example is that the rate of pressure decline during the pseudosteady state is the same throughout the drainage area. This means that the *average reservoir pressure,* $p_r$, is declining at the same rate of 1.719 psi, therefore the change in $p_r$ from 10 to 200 hours is:

$$\Delta\bar{p}_r = (1.719)(200 - 10) = 326.6 \text{ psi}$$

## Example 6-18

An oil well is producing under a constant bottom-hole flowing pressure of 1,500 psi. The current average reservoir pressure $p_r$ is 3,200 psi.

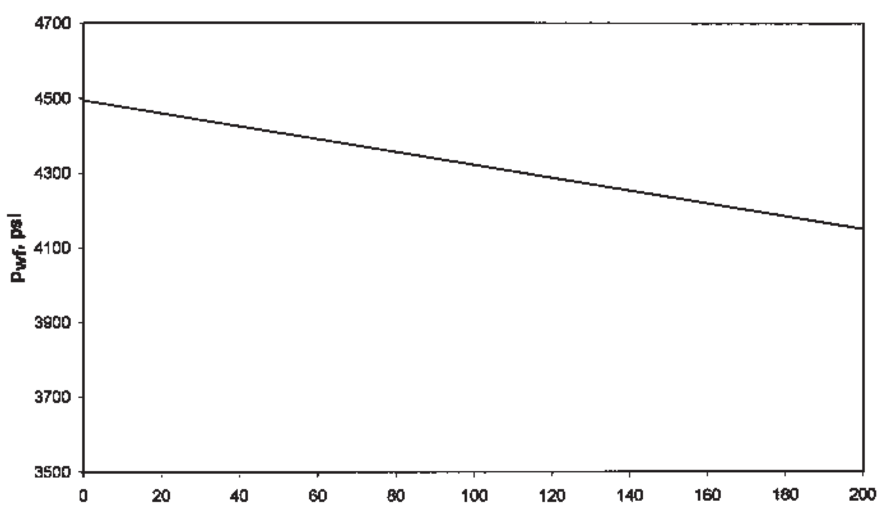



**Figure 6-25.** Bottom-hole flowing pressure as a function of time.

The well is developed in the center of a 40-acre square drilling pattern. Given the following additional information:

$\phi = 16\%$    $h = 15$ ft    $k = 50$ md
$\mu = 26$ cp    $B_o = 1.15$ bbl/STB    $c_t = 10 \times 10^{-6}$ psi$^{-1}$
$r_w = 0.25$ ft

calculate the flow rate.

## Solution

Because the volumetric average pressure is given, solve for the flow rate by applying Equation 6-138.

$$Q = \frac{(50)(15)(3200-1500)}{(162.6)(1.15)(2.6)\log\left[\dfrac{(4)(40)(43{,}560)}{1.781(30.8828)(0.25^2)}\right]}$$
$$= 416 \text{ STB/day}$$

## Radial Flow of Compressible Fluids (Gases)

The radial diffusivity equation as expressed by Equation 6-106 was developed to study the performance of compressible fluid under unsteady-state conditions. The equation has the following form:

$$\frac{\partial^2 m(p)}{\partial r^2} + \frac{1}{r}\frac{\partial m(p)}{\partial r} = \frac{\phi \mu c_t}{0.000264\,k}\frac{\partial m(p)}{\partial t}$$

For the semisteady-state flow, the rate of change of the real gas pseudopressure with respect to time is constant, i.e.,

$$\frac{\partial m(p)}{\partial t} = \text{constant}$$

Using the same technique identical to that described previously for liquids gives the following exact solution to the diffusivity equation:

$$Q_g = \frac{kh[m(\bar{p}_r) - m(p_{wf})]}{1422\,T\left[\ln\left(\dfrac{r_e}{r_w}\right) - 0.75\right]} \qquad (6\text{-}138)$$

where $Q_g$ = gas flow rate, Mscf/day
T = temperature, °R
k = permeability, md

Two approximations to the above solution are widely used. These approximations are:

- Pressure-squared approximation
- Pressure-approximation

## Pressure-Squared Approximation Method

As outlined previously, the method provides us with compatible results to that of the exact solution approach when p < 2,000. The solution has the following familiar form:

$$Q_g = \frac{kh(\bar{p}_r^2 - p_{wf}^2)}{1422\,T\bar{\mu}\,\bar{z}\left(\ln\frac{r_e}{r_w} - 0.75\right)} \tag{6-139}$$

The gas properties $\bar{z}$ and $\mu$ are evaluated at:

$$\bar{p} = \sqrt{\frac{(\bar{p}_r)^2 + p_{wf}^2}{2}}$$

## Pressure-Approximation Method

This approximation method is applicable at p > 3,000 psi and has the following mathematical form:

$$Q_g = \frac{kh(\bar{p}_r - p_{wf})}{1422\,\bar{\mu}\bar{B}_g\left(\ln\frac{r_e}{r_w} - 0.75\right)} \tag{6-140}$$

with the gas properties evaluated at:

$$\bar{p} = \frac{\bar{p}_r + p_{wf}}{2}$$

where $Q_g$ = gas flow rate, Mscf/day
$k$ = permeability, md
$\bar{B}_g$ = gas formation volume factor at average pressure, bbl/scf

The gas formation volume factor is given by the following expression:

$$\bar{B}_g = 0.00504 \frac{\bar{z}\, T}{\bar{p}}$$

In deriving the flow equations, the following two main assumptions were made:

- Uniform permeability throughout the drainage area
- Laminar (viscous) flow

Before using any of the previous mathematical solutions to the flow equations, the solution must be modified to account for the possible deviation from the above two assumptions. Introducing the following two correction factors into the solution of the flow equation can eliminate the above two assumptions:

- Skin factor
- Turbulent flow factor

## Skin Factor

It is not unusual for materials such as mud filtrate, cement slurry, or clay particles to enter the formation during drilling, completion, or workover operations and reduce the permeability around the wellbore. This effect is commonly referred to as a *wellbore damage* and the region of altered permeability is called the *skin zone.* This zone can extend from a few inches to several feet from the wellbore. Many other wells are stimulated by acidizing or fracturing, which in effect increase the permeability near the wellbore. Thus, the permeability near the wellbore is always different from the permeability away from the well where the formation has not been affected by drilling or stimulation. A schematic illustration of the skin zone is shown in Figure 6-26.

Those factors that cause damage to the formation can produce additional localized pressure drop during flow. This additional pressure drop is commonly referred to as $\Delta p_{skin}$. On the other hand, well stimulation techniques will normally enhance the properties of the formation and increase the permeability around the wellbore, so that a decrease in pressure

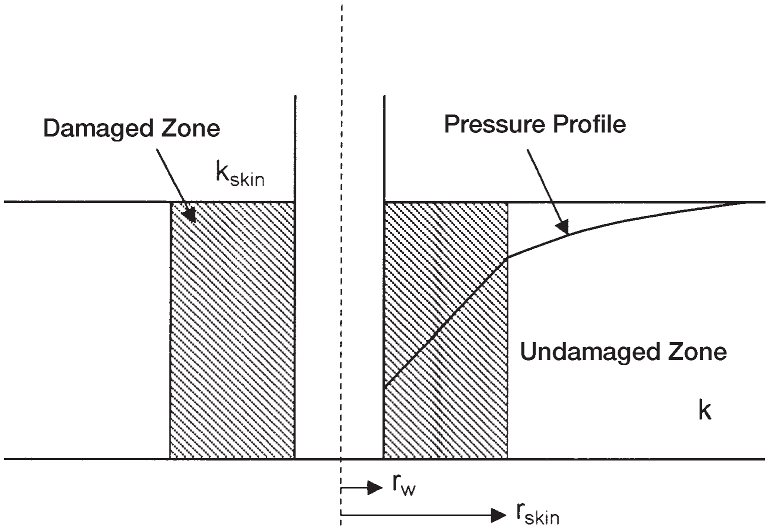



**Figure 6-26.** Near wellbore skin effect.

drop is observed. The resulting effect of altering the permeability around the well bore is called the *skin effect.*

Figure 6-27 compares the differences in the skin zone pressure drop for three possible outcomes:

- First Outcome:
  $\Delta p_{skin} > 0$, indicates an additional pressure drop due to wellbore damage, i.e., $k_{skin} < k$.
- Second Outcome:
  $\Delta p_{skin} < 0$, indicates less pressure drop due to wellbore improvement, i.e., $k_{skin} > k$.
- Third Outcome:
  $\Delta p_{skin} = 0$, indicates no changes in the wellbore condition, i.e., $k_{skin} = k$.

Hawkins (1956) suggested that the permeability in the skin zone, i.e., $k_{skin}$, is uniform and the pressure drop across the zone can be approximated by Darcy's equation. Hawkins proposed the following approach:

$$\Delta p_{skin} = \begin{bmatrix} \Delta p \text{ in skin zone} \\ \text{due to } k_{skin} \end{bmatrix} - \begin{bmatrix} \Delta p \text{ in the skin zone} \\ \text{due to } k \end{bmatrix}$$

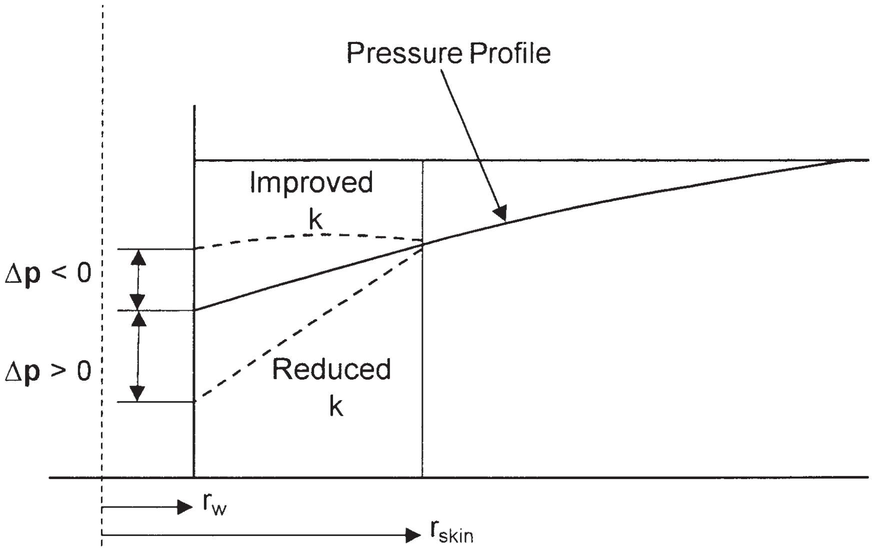



**Figure 6-27.** Representation of positive and negative skin effects.

Applying Darcy's equation gives:

$$\Delta p_{skin} = \left[\frac{Q_o B_o \mu_o}{0.00708\, h\, k_{skin}}\right] \ln\left(\frac{r_{skin}}{r_w}\right) - \left[\frac{Q_o B_o \mu_o}{0.00708\, h\, k}\right] \ln\left(\frac{r_{skin}}{r_w}\right)$$

or

$$\Delta p_{skin} = \left(\frac{Q_o B_o \mu_o}{0.00708\, kh}\right)\left[\frac{k}{k_{skin}} - 1\right] \ln\left(\frac{r_{skin}}{r_w}\right)$$

where $k$ = permeability of the formation, md
$k_{skin}$ = permeability of the skin zone, md

The above expression for determining the additional pressure drop in the skin zone is commonly expressed in the following form:

$$\Delta p_{skin} = \left[\frac{Q_o B_o \mu_o}{0.00708\, kh}\right] s = 141.2\left[\frac{Q_o B_o \mu_o}{kh}\right] s \qquad (6\text{-}141)$$

where s is called the skin factor and defined as:

$$s = \left[\frac{k}{k_{skin}} - 1\right] \ln\left(\frac{r_{skin}}{r_w}\right) \quad (6\text{-}142)$$

Equation 6-143 provides some insight into the physical significance of the sign of the skin factor. There are only three possible outcomes in evaluating the skin factor s:

- **Positive Skin Factor, s > 0**
  When a damaged zone near the wellbore exists, $k_{skin}$ is less than k and hence s is a positive number. The magnitude of the skin factor increases as $k_{skin}$ decreases and as the depth of the damage $r_{skin}$ increases.
- **Negative Skin Factor, s < 0**
  When the permeability around the well $k_{skin}$ is higher than that of the formation k, a negative skin factor exists. This negative factor indicates an improved wellbore condition.
- **Zero Skin Factor, s = 0**
  Zero skin factor occurs when no alternation in the permeability around the wellbore is observed, i.e., $k_{skin} = k$.

Equation 6-143 indicates that a negative skin factor will result in a negative value of $\Delta p_{skin}$. This implies that a stimulated well will require less pressure drawdown to produce at rate q than an equivalent well with uniform permeability.

The proposed modification of the previous flow equation is based on the concept that the actual total pressure drawdown will increase or decrease by an amount of $\Delta p_{skin}$. Assuming that $(\Delta p)_{ideal}$ represents the pressure drawdown for a drainage area with a uniform permeability k, then:

$$(\Delta p)_{actual} = (\Delta p)_{ideal} + (\Delta p)_{skin}$$

or

$$(p_i - p_{wf})_{actual} = (p_i - p_{wf})_{ideal} + \Delta p_{skin} \quad (6\text{-}143)$$

The above concept as expressed by Equation 6-144 can be applied to all the previous flow regimes to account for the skin zone around the wellbore as follows:

## Steady-State Radial Flow

Substituting Equations 6-27 and 6-142 into Equation 6-144 gives:

$$(p_i - p_{wf})_{actual} = \left[\frac{Q_o B_o \mu_o}{0.00708\, kh}\right] \ln\left(\frac{r_e}{r_w}\right) + \left[\frac{Q_o B_o \mu_o}{0.00708\, kh}\right] s$$

or

$$Q_o = \frac{0.00708\, kh\,(p_i - p_{wf})}{\mu_o B_o \left[\ln \frac{r_e}{r_w} + s\right]} \tag{6-144}$$

where $Q_o$ = oil flow rate, STB/day
$k$ = permeability, md
$h$ = thickness, ft
$s$ = skin factor
$B_o$ = oil formation volume factor, bbl/STB
$\mu_o$ = oil viscosity, cp
$p_i$ = initial reservoir pressure, psi
$p_{wf}$ = bottom hole flowing pressure, psi

## Unsteady-State Radial Flow

- **For Slightly Compressible Fluids:**
  Combining Equations 6-83 and 6-142 with that of Equation 6-144 yields:

$$p_i - p_{wf} = 162.6\left(\frac{Q_o B_o \mu_o}{kh}\right)\left[\log \frac{kt}{\phi \mu c_t r_w^2} - 3.23\right] + 141.2\left(\frac{Q_o B_o \mu_o}{kh}\right) s$$

or

$$p_i - p_{wf} = 162.6\left(\frac{Q_o B_o \mu_o}{kh}\right)\left[\log \frac{kt}{\phi \mu c_t r_w^2} - 3.23 + 0.87 s\right] \tag{6-145}$$

- **For Compressible Fluids:**
  A similar approach to that of the above gives:

$$m(p_{wf}) = m(p_i) - \frac{1637\,Q_g\,T}{kh}\left[\log\frac{kt}{\phi\mu c_{ti} r_w^2} - 3.23 + 0.87s\right] \tag{6-146}$$

and, in terms of the pressure-squared approach, gives:

$$p_{wf}^2 = p_i^2 - \frac{1637\,Q_g\,T\,\bar{z}\,\bar{\mu}}{kh}\left[\log\frac{kt}{\phi\mu_i c_{ti} r_w^2} - 3.23 + 0.87s\right] \tag{6-147}$$

## Pseudosteady-State Flow

- **For Slightly Compressible Fluids:**
  Introducing the skin factor into Equation 6-135 gives:

$$Q_o = \frac{0.00708\,kh(\bar{p}_r - p_{wf})}{\mu_o B_o\left[\ln\left(\frac{r_e}{r_w}\right) - 0.75 + s\right]} \tag{6-148}$$

- **For Compressible Fluids:**

$$Q_g = \frac{kh[m(\bar{p}_r) - m(P_{wf})]}{1422\,T\left[\ln\left(\frac{r_e}{r_w}\right) - 0.75 + s\right]} \tag{6-149}$$

or, in terms of the pressure-squared approximation, gives:

$$Q_g = \frac{kh(p_r^2 - p_{wf}^2)}{1422\,T\,\bar{\mu}\,\bar{z}\left[\ln\left(\frac{r_e}{r_w}\right) - 0.75 + s\right]} \tag{6-150}$$

where $Q_g$ = gas flow rate, Mscf/day
k = permeability, md
T = temperature, °R

$(\overline{\mu}_g)$ = gas viscosity at average pressure $\overline{p}$, cp
$\overline{z}_g$ = gas compressibility factor at average pressure $\overline{p}$

## Example 6-19

Calculate the skin factor resulting from the invasion of the drilling fluid to a radius of 2 feet. The permeability of the skin zone is estimated at 20 md as compared with the unaffected formation permeability of 60 md. The wellbore radius is 0.25 ft.

## Solution

Apply Equation 6-143 to calculate the skin factor:

$$s = \left[\frac{60}{20} - 1\right] \ln\left(\frac{2}{0.25}\right) = 4.16$$

Matthews and Russell (1967) proposed an alternative treatment to the skin effect by introducing the **effective** or **apparent wellbore radius** $r_{wa}$ that accounts for the pressure drop in the skin. They define $r_{wa}$ by the following equation:

$$r_{wa} = r_w e^{-s} \tag{6-151}$$

All of the ideal radial flow equations can be also modified for the skin by simply replacing wellbore radius $r_w$ with that of the apparent wellbore radius $r_{wa}$. For example, Equation 6-146 can be equivalently expressed as:

$$p_i - p_{wf} = 162.6\left(\frac{Q_o B_o \mu_o}{kh}\right)\left[\log \frac{kt}{\phi \mu_o c_t r_{wa}^2} - 3.23\right] \tag{6-152}$$

## Turbulent Flow Factor

All of the mathematical formulations presented so far are based on the assumption that laminar flow conditions are observed during flow. During radial flow, the flow velocity increases as the wellbore is approached. This increase in the velocity might cause the development of a turbulent flow around the wellbore. If turbulent flow does exist, it is most likely to occur with gases and causes an additional pressure drop similar to that caused by the skin effect. The term *non-Darcy flow* has been adopted by the industry to describe the additional pressure drop due to the turbulent (non-Darcy) flow.

Referring to the additional real gas pseudopressure drop due to non-Darcy flow as $\Delta\psi$ non-Darcy, the total (actual) drop is given by:

$$(\Delta\psi)_{actual} = (\Delta\psi)_{ideal} + (\Delta\psi)_{skin} + (\Delta\psi)_{non\text{-}Darcy}$$

Wattenburger and Ramey (1968) proposed the following expression for calculating $(\Delta\psi)_{non\text{-}Darcy}$:

$$(\Delta\psi)_{non\text{-}Darcy} = 3.161 \times 10^{-12} \left[ \frac{\beta T \gamma_g}{\mu_{gw} h^2 r_w} \right] Q_g^2 \quad (6\text{-}153)$$

The above equation can be expressed in a more convenient form as:

$$(\Delta\psi)_{non\text{-}Darcy} = F Q_g^2 \quad (6\text{-}154)$$

where F is called the *non-Darcy flow coefficient* and is given by:

$$F = 3.161 \times 10^{-12} \left[ \frac{\beta T \gamma_g}{\mu_{gw} h^2 r_w} \right] \quad (6\text{-}155)$$

where $Q_g$ = gas flow rate, Mscf/day
$\mu_{gw}$ = gas viscosity as evaluated at $p_{wf}$, cp
$\gamma_g$ = gas specific gravity
h = thickness, ft
F = non-Darcy flow coefficient, $psi^2/cp/(Mscf/day)^2$
$\beta$ = turbulence parameter

Jones (1987) proposed a mathematical expression for estimating the turbulence parameter $\beta$ as:

$$\beta = 1.88\,(10^{-10})\,(k)^{-1.47}\,(\phi)^{-0.53} \quad (6\text{-}156)$$

where k = permeability, md
$\phi$ = porosity, fraction

The term $FQ_g^2$ can be included in all the compressible gas flow equations in the same way as the skin factor. This non-Darcy term is interpreted as being a *rate-dependent skin.* The modification of the gas flow equations to account for the turbulent flow condition is given below:

## Unsteady-State Radial Flow

The gas flow equation for an unsteady-state flow is given by Equation 6-147 and can be modified to include the additional drop in the real gas potential as:

$$m(p_i) - m(p_{wf}) = \left(\frac{1637\, Q_g\, T}{kh}\right)\left[\log \frac{kt}{\phi \mu_i c_{ti} r_w^2} - 3.23 + 0.87s\right] + FQ_g^2 \quad (6\text{-}157)$$

Equation 6-158 is commonly written in a more convenient form as:

$$m(p_i) - m(p_{wf}) = \left(\frac{1637\, Q_g\, T}{kh}\right) \times \left[\log \frac{kt}{\phi \mu_i c_{ti} r_w^2} - 3.23 + 0.87s + 0.87\, DQ_g\right] \quad (6\text{-}158)$$

where the term $DQ_g$ is interpreted as the rate dependent skin factor. The coefficient D is called the **inertial** or **turbulent flow factor** and given by:

$$D = \frac{Fkh}{1422T} \quad (6\text{-}159)$$

The true skin factor s, which reflects the formation damage or stimulation, is usually combined with the non-Darcy rate dependent skin and labeled as the **apparent** or **total skin factor:**

$$s' = s + DQ_g \quad (6\text{-}160)$$

or

$$m(p_i) - m(p_{wf}) = \left[\frac{1637\, Q_g\, T}{kh}\right] \times \left[\log \frac{kt}{\phi \mu_i c_{ti} r_w^2} - 3.23 + 0.87s'\right] \quad (6\text{-}161)$$

Equation 6-162 can be expressed in the pressure-squared approximation form as:

$$p_i^2 - p_{wf}^2 = \left[\frac{1637\, Q_g\, T\, \bar{z}\, \bar{\mu}}{kh}\right]\left[\log \frac{kt}{\phi \mu_i\, c_{ti}\, r_w^2} - 3.23 + 0.87 s'\right] \qquad (6\text{-}162)$$

where $Q_g$ = gas flow rate, Mscf/day
$t$ = time, hr
$k$ = permeability, md
$\mu_i$ = gas viscosity as evaluated at $p_i$, cp

## Semisteady-State Flow

Equations 6-150 and 6-151 can be modified to account for the non-Darcy flow as follows:

$$Q_g = \frac{kh[m(\bar{p}_r) - m(p_{wf})]}{1422\, T\left[\ln\left(\frac{r_e}{r_w}\right) - 0.75 + s + DQ_g\right]} \qquad (6\text{-}163)$$

or in terms of the pressure-squared approach:

$$Q_g = \frac{kh(\bar{p}_r^2 - p_{wf}^2)}{1422\, T\, \bar{\mu}\, \bar{z}\left[\ln\left(\frac{r_e}{r_w}\right) - 0.75 + s + DQ_g\right]} \qquad (6\text{-}164)$$

where the coefficient D is defined as:

$$D = \frac{Fkh}{1422\, T} \qquad (6\text{-}165)$$

## Steady-State Flow

Similar to the above modification procedure, Equations 6-44 and 6-45 can be expressed as:

$$Q_g = \frac{kh[m(p_i) - m(p_{wf})]}{1422\,T\left[\ln\frac{r_e}{r_w} - 0.5 + s + DQ_g\right]} \tag{6-166}$$

$$Q_g = \frac{kh(p_e^2 - p_{wf}^2)}{1422\,T\,\bar{\mu}\,\bar{z}\left[\ln\frac{r_e}{r_w} - 0.5 + s + DQ_g\right]} \tag{6-167}$$

where D is defined by Equation 6-166.

## Example 6-20

A gas well has an estimated wellbore damage radius of 2 feet and an estimated reduced permeability of 30 md. The formation has a permeability and porosity of 55 md and 12%. The well is producing at a rate of 20 Mscf/day with a gas gravity of 0.6. The following additional data are available:

$r_w = 0.25$ $h = 20'$ $T = 140°F$ $\mu_{gw} = 0.013$ cp

Calculate the apparent skin factor.

## Solution

*Step 1.* Calculate the skin factor from Equation 6-143

$$s = \left[\frac{55}{30} - 1\right]\ln\left(\frac{2}{0.25}\right) = 1.732$$

*Step 2.* Calculate the turbulence parameter β by applying Equation 6-155:

$$\beta = 1.88\,(10)^{-10}\,(55)^{-1.47}\,(0.12)^{-0.53} = 159.904 \times 10^6$$

*Step 3.* Calculate the non-Darcy flow coefficient from Equation 6-156:

$$F = 3.1612 \times 10^{-12}\left[\frac{159.904 \times 10^6\,(600)\,(0.6)}{(0.013)\,(20)^2\,(0.25)}\right] = 0.14$$

*Step 4.* Calculate the coefficient D from Equation 6-160:

$$D = \frac{(0.14)(55)(20)}{(1422)(600)} = 1.805 \times 10^{-4}$$

*Step 5.* Estimate the apparent skin factor by applying Equation 6-161:

$$s' = 1.732 + (1.805 \times 10^{-4})(20{,}000) = 5.342$$

## PRINCIPLE OF SUPERPOSITION

The solutions to the radial diffusivity equation as presented earlier in this chapter appear to be applicable only for describing the pressure distribution in an infinite reservoir that was caused by a constant production from a single well. Since real reservoir systems usually have several wells that are operating at varying rates, a more generalized approach is needed to study the fluid flow behavior during the unsteady-state flow period.

The principle of superposition is a powerful concept that can be applied to remove the restrictions that have been imposed on various forms of solution to the transient flow equation. Mathematically the superposition theorem states that any sum of individual solutions to the diffusivity equation is also a solution to that equation. This concept can be applied to account for the following effects on the transient flow solution:

- Effects of multiple wells
- Effects of rate change
- Effects of the boundary
- Effects of pressure change

Slider (1976) presented an excellent review and discussion of the practical applications of the principle of superposition in solving a wide variety of unsteady-state flow problems.

### Effects of Multiple Wells

Frequently, it is desired to account for the effects of more than one well on the pressure at some point in the reservoir. The superposition concept states that the total pressure drop at any point in the reservoir is the sum of the pressure changes at that point caused by flow in each of the wells in the reservoir. In other words, we simply superimpose one effect upon the other.

Consider Figure 6-28, which shows three wells that are producing at different flow rates from an infinite acting reservoir, i.e., unsteady-state

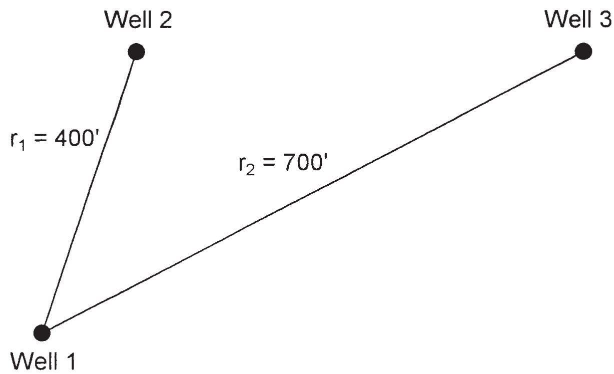



**Figure 6-28.** Well layout for Example 6-20.

flow reservoir. The principle of superposition shows that the total pressure drop observed at any well, e.g., Well 1, is:

$$(\Delta p)_{\text{total drop at well 1}} = (\Delta p)_{\text{drop due to well 1}} + (\Delta p)_{\text{drop due to well 2}} + (\Delta p)_{\text{drop due to well 3}}$$

The pressure drop at Well 1 due to its own production is given by the log-approximation to the $E_i$-function solution presented by Equation 6-146, or:

$$(p_i - p_{wf}) = (\Delta p)_{\text{well 1}} = \frac{162.6 Q_{o1} B_o \mu_o}{kh} \times \left[\log\left(\frac{kt}{\phi \mu c_t r_w^2}\right) - 3.23 + 0.87s\right]$$

where t = time, hr
s = skin factor
k = permeability, md
$Q_{o1}$ = oil flow rate from Well 1

The pressure drop at Well 1 due to production at Wells 2 and 3 must be written in terms of the $E_i$-function solution as expressed by Equation 6-78. The log-approximation cannot be used because we are calculating the pressure at a large distance r from the well, i.e., the argument x > 0.01, or:

$$(p_i - p_{wf})_{\text{total at well 1}} = \left(\frac{162.6 Q_{o1} B_o \mu_o}{kh}\right)$$

$$\times \left[\log\left(\frac{kt}{\phi \mu c_t r_w^2}\right) - 3.23 + 0.87s\right] - \left(\frac{70.6\, Q_{o2} B_o \mu_o}{kh}\right)$$

$$\times E_i\left[-\frac{948 \phi \mu c_t r_1^2}{kt}\right] - \left(\frac{70.6\, Q_{o3} B_o \mu_o}{kh}\right) E_i\left[-\frac{948 \phi \mu c_t r_2^2}{kt}\right]$$

where $Q_{o1}$, $Q_{o2}$, and $Q_{o3}$ refer to the respective producing rates of Wells 1, 2, and 3.

The above computational approach can be used to calculate the pressure at Wells 2 and 3. Further, it can be extended to include any number of wells flowing under the unsteady-state flow condition. It should also be noted that if the point of interest is an operating well, the skin factor s must be included for that well only.

## Example 6-21

Assume that the three wells as shown in Figure 6-28 are producing under a transient flow condition for 15 hours. The following additional data are available:

| | |
|---|---|
| $Q_{o1} = 100$ STB/day | $h = 20'$ |
| $Q_{o2} = 160$ STB/day | $\phi = 15\%$ |
| $Q_{o3} = 200$ STB/day | $k = 40$ md |
| $p_i = 4500$ psi | $r_w = 0.25'$ |
| $B_o = 1.20$ bbl/STB | $\mu_o = 2.0$ cp |
| $c_t = 20 \times 10^{-6}$ psi$^{-1}$ | $r_1 = 400'$ |
| $(s)_{\text{well 1}} = -0.5$ | $r_2 = 700'$ |

If the three wells are producing at a constant flow rate, calculate the sand face flowing pressure at Well 1.

## Solution

*Step 1.* Calculate the pressure drop at Well 1 caused by its own production by using Equation 6-146.

$$(\Delta p)_{\text{well 1}} = \frac{(162.6)(100)(1.2)(2.0)}{(40)(20)}$$

$$\times \left[\log\left(\frac{(40)(15)}{(0.15)(2)(20\times 10^{-6})(0.25)^2}\right) - 3.23 + 0.87\ (0)\right]$$

$$= 270.2 \text{ psi}$$

*Step 2.* Calculate the pressure drop at Well 1 due to the production from Well 2.

$$(\Delta p)_{\text{due to well 2}} = -\frac{(70.6)(160)(1.2)(2)}{(40)(20)}$$

$$\times E_i\left[-\frac{(948)(0.15)(2.0)(20\times 10^{-6})(400)^2}{(40)(15)}\right]$$

$$= 33.888\ [-E_i\ (-1.5168)]$$

$$= (33.888)\ (0.13) = 4.41 \text{ psi}$$

*Step 3.* Calculate pressure drop due to production from Well 3.

$$(\Delta p)_{\text{due to well 3}} = -\frac{(70.6)(200)(1.2)(2)}{(40)(20)}$$

$$\times E_i\left[-\frac{(948)(0.15)(2.0)(20\times 10^{-6})(700)^2}{(40)(15)}\right]$$

$$= (42.36)[-E_i\ (-4.645)]$$

$$= (42.36)(1.84\times 10^{-3}) = 0.08 \text{ psi}$$

*Step 4.* Calculate total pressure drop at Well 1.

$$(\Delta p)_{\text{total at well 1}} = 270.2 + 4.41 + 0.08 = 274.69 \text{ psi}$$

*Step 5.* Calculate $p_{wf}$ at Well 1.

$$p_{wf} = 4500 - 274.69 = 4225.31 \text{ psi}$$

## Effects of Variable Flow Rates

All of the mathematical expressions presented previously in this chapter require that the wells produce at a constant rate during the transient flow periods. Practically all wells produce at varying rates and, therefore, it is important that we be able to predict the pressure behavior when the rate changes. For this purpose, the concept of superposition states, "**Every flow rate change in a well will result in a pressure response which is independent of the pressure responses caused by other previous rate changes.**" Accordingly, the total pressure drop that has occurred at any time is the summation of pressure changes caused separately by each *net* flow rate change.

Consider the case of a shut-in well, i.e., Q = 0, that was then allowed to produce at a series of constant rates for the different time periods shown in Figure 6-29. To calculate the total pressure drop at the sand face at time $t_4$, the composite solution is obtained by adding the individual constant-rate solutions at the specified rate-time sequence, or:

$$(\Delta p)_{total} = (\Delta p)_{\text{due to } (Q_{o1} - 0)} + (\Delta p)_{\text{due to } (Q_{o2} - Q_{o1})} + (\Delta p)_{\text{due to } (Q_{o3} - Q_{o2})} + (\Delta p)_{\text{due to } (Q_{o4} - Q_{o3})}$$

The above expression indicates that there are four contributions to the total pressure drop resulting from the four individual flow rates.

The first contribution results from increasing the rate from 0 to $Q_1$ and is in effect over the entire time period $t_4$, thus:

$$(\Delta p)_{Q1-o} = \left[\frac{162.6\,(Q_1 - 0)\,B\mu}{kh}\right]\left[\log\left(\frac{kt_4}{\phi\mu c_t r_w^2}\right) - 3.23 + 0.87s\right]$$

It is essential to notice the *change* in the rate, i.e., (new rate – old rate), that is used in the above equation. It is the change in the rate that causes the pressure disturbance. Further, it should be noted that the "time" in the equation represents the *total elapsed time* since the change in the rate has been in effect.

Second contribution results from decreasing the rate from $Q_1$ to $Q_2$ at $t_1$, thus:

$$(\Delta p)_{Q_2 - Q_1} = \left[\frac{162.6\,(Q_2 - Q_1)\,B\mu}{kh}\right] \times \left[\log\left(\frac{k(t_4 - t_1)}{\phi\mu c_t r_w^2}\right) - 3.23 + 0.87s\right]$$

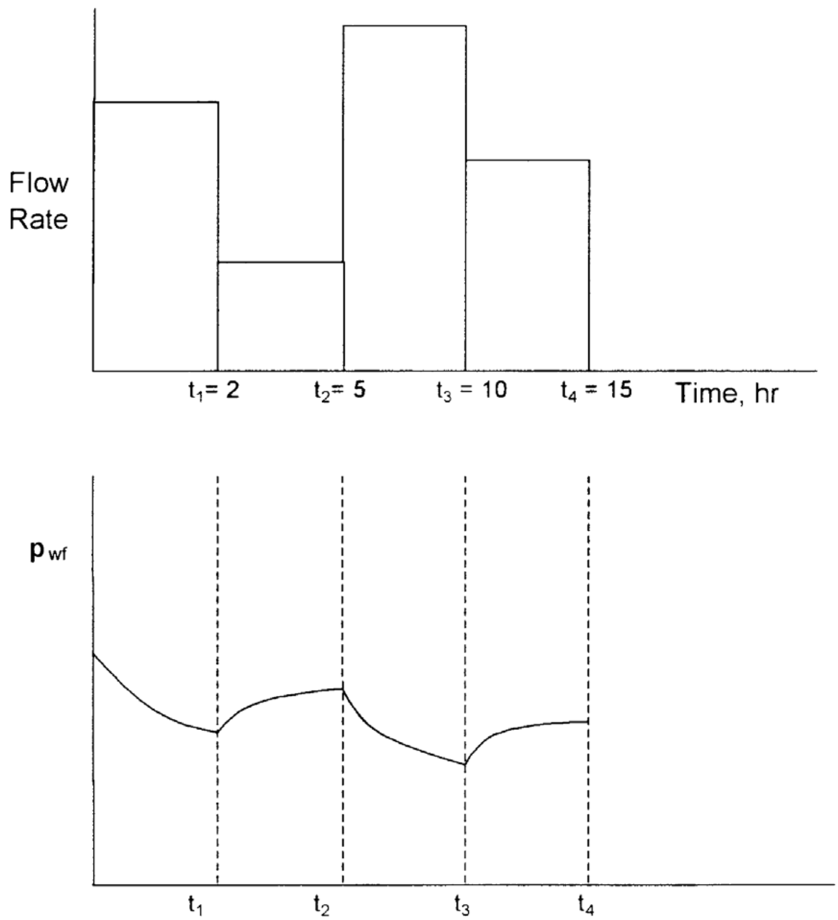



**Figure 6-29.** Production and pressure history of a well.

Using the same concept, the contributions from $Q_2$ to $Q_3$ and from $Q_3$ to $Q_4$ can be computed as:

$$(\Delta p)_{Q3-Q2} = \left[\frac{162.6\,(Q_3 - Q_2)\,B\mu}{kh}\right] \times \left[\log\left(\frac{k\,(t_4 - t_2)}{\phi\mu c_t r_w^2}\right) - 3.23 + 0.87s\right]$$

$$(\Delta p)_{Q_4 - Q_3} = \left[\frac{162.6\,(Q_4 - Q_3)\,B\mu}{kh}\right]$$

$$\times\left[\log\left(\frac{k\,(t_4 - t_3)}{\phi\mu\, c_t\, r_w^2}\right) - 3.23 + 0.87s\right]$$

The above approach can be extended to model a well with several rate changes. Note, however, the above approach is valid only if the well is flowing under the unsteady-state flow condition for the total time elapsed since the well began to flow at its initial rate.

## Example 6-22

Figure 6-29 shows the rate history of a well that is producing under transient flow condition for 15 hours. Given the following data:

| | |
|---|---|
| $p_i = 5000$ psi | $h = 20'$ |
| $B_o = 1.1$ bbl/STB | $\phi = 15\%$ |
| $\mu_o = 2.5$ cp | $r_w = 0.3'$ |
| $c_t = 20 \times 10^{-6}$ psi$^{-1}$ | $s = 0$ |
| $k = 40$ md | |

calculate the sand face pressure after 15 hours.

## Solution

*Step 1.* Calculate the pressure drop due to the first flow rate for the entire flow period.

$$(\Delta p)_{Q_1 - 0} = \frac{(162.6)\,(100 - 0)\,(1.1)\,(2.5)}{(40)\,(20)}$$

$$\times\left[\log\left[\frac{(40)\,(15)}{(0.15)\,(2.5)\,(20 \times 10^{-6})\,(0.3)^2}\right] - 3.23 + 0\right] = 319.6\,\text{psi}$$

*Step 2.* Calculate the additional pressure change due to the change of the flow rate from 100 to 70 STB/day.

$$(\Delta p)_{Q_2 - Q_1} = \frac{(162.6)(70 - 100)(1.1)(2.5)}{(40)(20)}$$

$$\times \left[ \log \left[ \frac{(40)(15 - 2)}{(0.15)(2.5)(20 \times 10^{-6})(0.3)^2} \right] - 3.23 \right] = -94.85 \text{ psi}$$

*Step 3.* Calculate the additional pressure change due to the change of the flow rate from 70 to 150 STB/day.

$$(\Delta p)_{Q_3 - Q_2} = \frac{(162.6)(150 - 70)(1.1)(2.5)}{(40)(20)}$$

$$\times \left[ \log \left[ \frac{(40)(15 - 5)}{(0.15)(2.5)(20 \times 10^{-6})(0.3)^2} \right] - 3.23 \right] = 249.18 \text{ psi}$$

*Step 4.* Calculate the additional pressure change due to the change of the flow rate from 150 to 85 STB/day.

$$(\Delta p)_{Q_4 - Q_3} = \frac{(162.6)(85 - 150)(1.1)(2.5)}{(40)(20)}$$

$$\times \left[ \log \left[ \frac{(40)(15 - 10)}{(0.15)(2.5)(20 \times 10^{-6})(0.3)^2} \right] - 3.23 \right] = -190.44 \text{ psi}$$

*Step 5.* Calculate the total pressure drop:

$$(\Delta p)_{total} = 319.6 + (-94.85) + 249.18 + (-190.44) = 283.49 \text{ psi}$$

*Step 6.* Calculate wellbore pressure after 15 hours of transient flow:

$$p_{wf} = 5000 - 283.49 = 4716.51 \text{ psi}$$

## Effects of the Reservoir Boundary

The superposition theorem can also be extended to predict the pressure of a well in a bounded reservoir. Consider Figure 6-30, which shows a well that is located a distance r from the no-flow boundary, e.g., sealing fault.

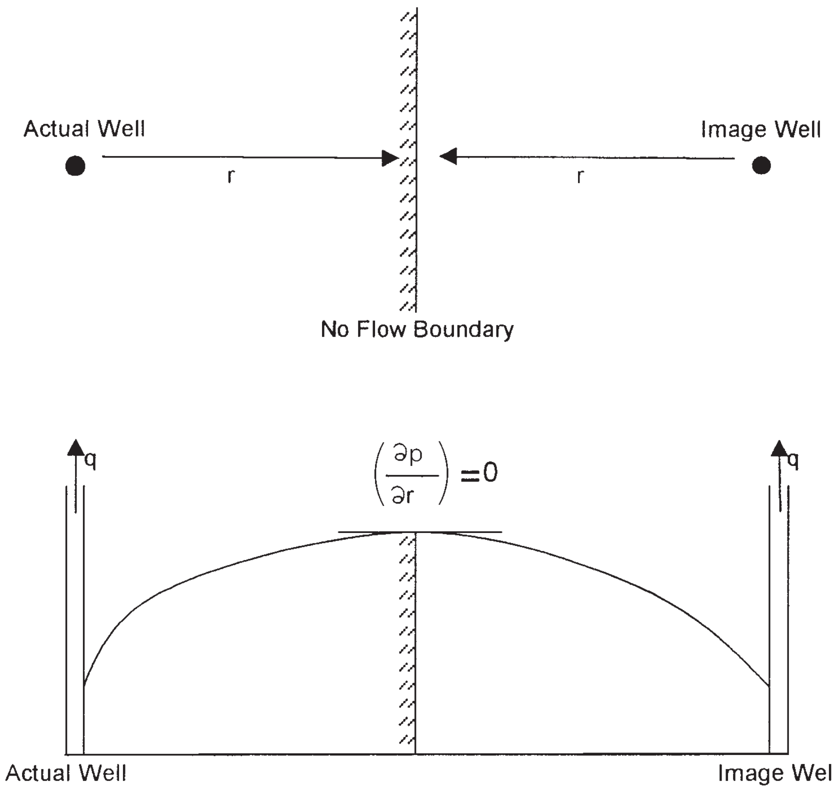



**Figure 6-30.** Method of images in solving boundary problems.

The no-flow boundary can be represented by the following pressure gradient expression:

$$\left(\frac{\partial p}{\partial r}\right)_{\text{Boundary}} = 0$$

Mathematically, the above boundary condition can be met by placing an *image* well, identical to that of the actual well, on the other side of the fault at exactly distance r. Consequently, the effect of the boundary on the pressure behavior of a well would be the same as the effect from an image well located a distance 2r from the actual well.

In accounting for the boundary effects, the superposition method is frequently called the ***method of images.*** Thus, for the problem of the system configuration given in Figure 6-30, the problem reduces to one of determining the effect of the image well on the actual well. The total pressure drop at the actual well will be the pressure drop due to its own production plus the additional pressure drop caused by an identical well at a distance of 2r, or:

$$(\Delta p)_{total} = (\Delta p)_{actual\ well} + (\Delta p)_{due\ to\ image\ well}$$

or

$$(\Delta p)_{total} = \frac{162.6\, Q_o\, B_o \mu_o}{kh}\left[\log\left(\frac{kt}{\phi \mu_o\, c_t\, r_w^2}\right) - 3.23 + 0.87s\right] - \left(\frac{70.6\, Q_o\, B_o\, \mu_o}{kh}\right) E_i\left(-\frac{948\, \phi \mu_o\, c_t\, (2r)^2}{kt}\right) \qquad (6\text{-}168)$$

Notice that this equation assumes the reservoir is infinite except for the indicated boundary. The effect of boundaries is always to cause greater pressure drop than those calculated for infinite reservoirs.

The concept of image wells can be extended to generate the pressure behavior of a well located within a variety of boundary configurations.

## Example 6-23

Figure 6-31 shows a well located between two sealing faults at 200 and 100 feet from the two faults. The well is producing under a transient flow condition at a constant flow rate of 200 STB/day.

Given:

| | |
|---|---|
| $p_i = 500$ psi | $k = 600$ md |
| $B_o = 1.1$ bbl/STB | $\phi = 17\%$ |
| $\mu_o = 2.0$ cp | $h = 25$ ft |
| $r_w = 0.3$ ft | $s = 0$ |
| $c_t = 25 \times 10^{-6}$ psi$^{-1}$ | |

calculate the sand face pressure after 10 hours.

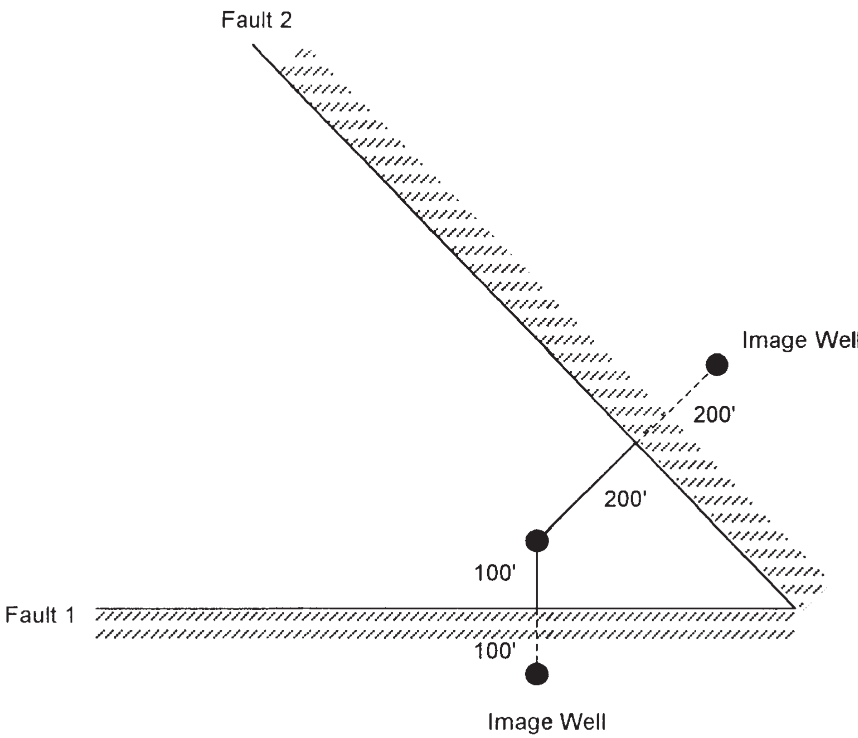



**Figure 6-31.** Well layout for Example 6-31.

## Solution

*Step 1.* Calculate the pressure drop due to the actual well flow rate.

$$(\Delta p)_{actual} = \frac{(162.6)(200)(1.1)(2.0)}{(60)(25)}$$

$$\times \left[\log\left[\frac{(60)(10)}{(0.17)(2)(0.17)(2)(25\times10^{-6})(0.3)^2}\right] - 3.23 + 0\right]$$

$$= 270.17$$

*Step 2.* Determine the additional pressure drop due to the first fault (i.e., image well 1):

$$(\Delta p)_{\text{image well 1}} = -\frac{(70.6)(200)(1.1)(2.0)}{(60)(25)}$$
$$\times E_i\left[-\frac{(948)(0.17)(2)(25\times10^{-6})(2\times100)^2}{(6)(10)}\right]$$
$$= 20.71\left[-E_i(-0.537)\right] = 10.64\text{ psi}$$

*Step 3.* Calculate the effect of the second fault (i.e., image well 2):

$$(\Delta p)_{\text{image well 2}} = 20.71\left[-E_i\left(\frac{-948(0.17)(2)(25\times10^{-6})(2\times200)^2}{(60)(10)}\right)\right]$$
$$= 20.71[-E_i(-2.15)] = 1.0\text{ psi}$$

*Step 4.* Total pressure drop is:

$$(\Delta p)_{\text{total}} = 270.17 + 10.64 + 1.0 = 281.8\text{ psi}$$

*Step 5.* $p_{wf} = 5000 - 281.8 = 4718.2$ psi

### Accounting for Pressure-Change Effects

Superposition is also used in applying the constant-pressure case. Pressure changes are accounted for in this solution in much the same way that rate changes are accounted for in the constant rate case. The description of the superposition method to account for the pressure-change effect is fully described in the Water Influx section in this book.

## TRANSIENT WELL TESTING

Detailed reservoir information is essential to the petroleum engineer in order to analyze the current behavior and future performance of the reservoir. Pressure transient testing is designed to provide the engineer with a quantitative analysis of the reservoir properties. A transient test is essentially conducted by creating a pressure disturbance in the reservoir and recording the pressure response at the wellbore, i.e., bottom-hole flowing pressure $p_{wf}$, as a function of time. The pressure transient tests most commonly used in the petroleum industry include:

- Pressure drawdown
- Pressure buildup
- Multirate
- Interference
- Pulse
- Drill stem
- Fall off
- Injectivity
- Step rate

It has long been recognized that the pressure behavior of a reservoir following a rate change directly reflects the geometry and flow properties of the reservoir. Information available from a well test includes:

- Effective permeability
- Formation damage or stimulation
- Flow barriers and fluid contacts
- Volumetric average reservoir pressure
- Drainage pore volume
- Detection, length, capacity of fractures
- Communication between wells

Only the drawdown and buildup tests are briefly described in the following two sections. There are several excellent books that comprehensively address the subject of well testing, notably:

- John Lee, *Well Testing* (1982)
- C. S. Matthews and D. G. Russell, *Pressure Buildup and Flow Test in Wells* (1967)
- Robert Earlougher, *Advances in Well Test Analysis* (1977)
- Canadian Energy Resources Conservation Board, *Theory and Practice of the Testing of Gas Wells* (1975)
- Roland Horn, *Modern Well Test Analysis* (1995)

## Drawdown Test

A pressure drawdown test is simply a series of bottom-hole pressure measurements made during a period of flow at constant producing rate.

Usually the well is shut-in prior to the flow test for a period of time sufficient to allow the pressure to equalize throughout the formation, i.e., to reach static pressure. A schematic of the ideal flow rate and pressure history is illustrated by Figure 6-32.

The fundamental objectives of drawdown testing are to obtain the average permeability, k, of the reservoir rock *within the drainage area of the well* and to assess the degree of damage of stimulation induced in the vicinity of the wellbore through drilling and completion practices. Other objectives are to determine the pore volume and to detect reservoir inhomogeneities *within the drainage area of the well.*

During flow at a constant rate of $Q_o$, the pressure behavior of a well in an infinite-acting reservoir (i.e., during the unsteady-state flow period) is given by Equation 6-146, as:

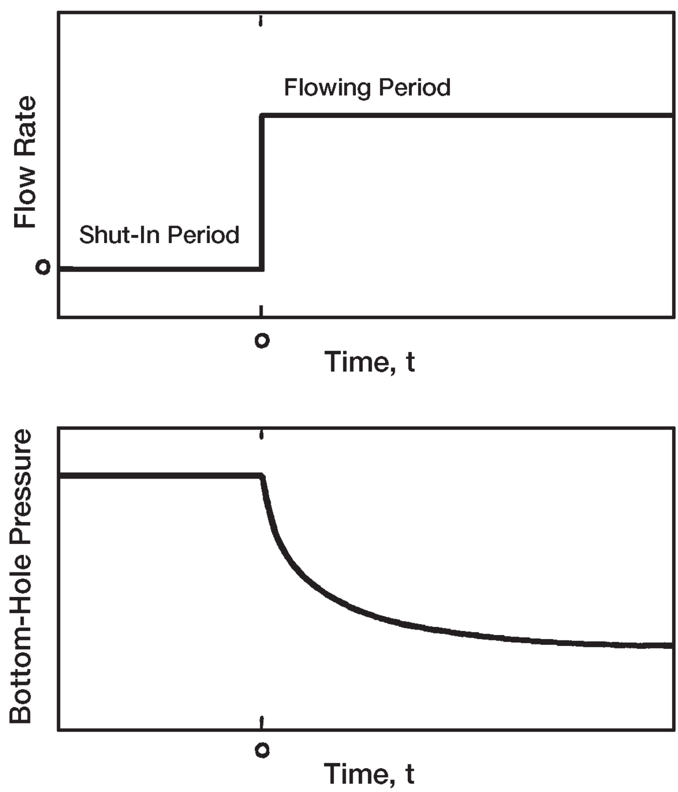



**Figure 6-32.** Idealized drawdown test.

$$p_{wf} = p_i - \frac{162.6\, Q_o\, B_o\, \mu}{kh}\left[\log\left(\frac{kt}{\phi\, \mu\, c_t\, r_w^2}\right) - 3.23 + 0.87\, s\right]$$

where k = permeability, md
t = time, hr
$r_w$ = wellbore radius
s = skin factor

The above expression can be written as:

$$p_{wf} = p_i - \frac{162.6\, Q_o\, B_o\, \mu}{kh} \times \left[\log(t) + \log\left(\frac{k}{\phi\, \mu\, c_t\, r_w^2}\right) - 3.23 + 0.87\, s\right] \quad (6\text{-}169)$$

Equation 6-170 is essentially an equation of a straight line and can be expressed as:

$$p_{wf} = a + m \log(t) \quad (6\text{-}170)$$

where

$$a = p_i - \frac{162.6\, Q_o\, B_o\, \mu}{kh}\left[\log\left(\frac{k}{\phi\, \mu\, c_t\, r_w^2}\right) - 3.23 + 0.87\, s\right]$$

The slope m is given by:

$$m = \frac{-162.6\, Q_o\, B_o\, \mu_o}{kh} \quad (6\text{-}171)$$

Equation 6-171 suggests that a plot of $p_{wf}$ versus time t on semilog graph paper would yield a straight line with a slope m in psi/cycle. This semilog straight-line relationship is illustrated by Figure 6-33.

Equation 6-172 can be also rearranged for the capacity kh of the drainage area of the well. If the thickness is known, then the average permeability is given by:

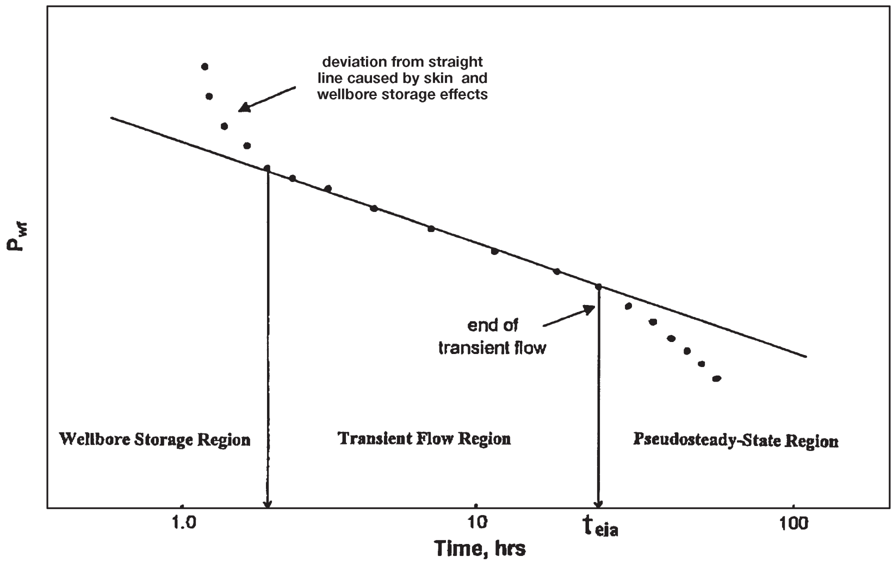



**Figure 6-33.** Semilog plot of pressure drawdown data.

$$k = \frac{-162.6\,Q_o\,B_o\,\mu_o}{mh} \tag{6-172}$$

where k = average permeability, md
m = slope, psi/cycle. Notice, slope m is negative.

Clearly, kh/μ or k/μ may be also estimated.

The skin effect can be obtained by rearranging Equation 6-170, as:

$$s = 1.151\left(\frac{p_{wf} - p_i}{m} - \log t - \log\frac{k}{\phi\mu\,c_t\,r_w^2} + 3.23\right)$$

or, more conveniently, if $p_{wf} = p_{1\,hr}$, which is found on the extension of the straight line at log t (1 hour), then:

$$s = 1.151\left(\frac{p_{1\,hr} - p_i}{m} - \log\frac{k}{\phi\mu\,c_t\,r_w^2} + 3.23\right) \tag{6-173}$$

In Equation 6-174, $p_{1\,hr}$ must be from the semilog straight line. If pressure data measured at 1 hour do not fall on that line, the line *must be extrapolated* to 1 hour and the extrapolated value of $p_{1\,hr}$ must be used in Equation 6-174. This procedure is necessary to avoid calculating an incorrect skin by using a wellbore-storage-influenced pressure. Figure 6-33 illustrates the extrapolation to $p_{1\,hr}$.

If the drawdown test is long enough, bottom-hole pressure will deviate from the semilog straight line and make the transition from infinite-acting to pseudosteady state.

It should be pointed out that the pressure drop due to the skin, as expressed by Equation 6-142, can be written in terms of the transient flow slope, m, by combining the equations:

$$m = 162.6 \frac{Q_o B_o \mu_o}{k h}$$

$$\Delta Ps = 141.2 \left[ \frac{Q_o B_o \mu_o}{kh} \right] s$$

Combining the two expressions gives

$$\Delta p_s = 0.87 \, m \, s$$

## Example 6-24[2]

Estimate oil permeability and skin factor from the drawdown data of Figure 6-34.

The following reservoir data are available:

| | |
|---|---|
| $h = 130$ ft | $\phi = 20$ percent |
| $r_w = 0.25$ ft | $p_i = 1154$ psi |
| $Q_o = 348$ STB/D | $m = -22$ psi/cycle |
| $B_o = 1.14$ bbl/STB | |
| $\mu_o = 3.93$ cp | |
| $c_t = 8.74 \times 10^{-6}$ psi$^{-1}$ | |

Assuming that the wellbore storage effects are not significant, calculate:

- Permeability
- Skin factor

[2]This example problem and the solution procedure are given by Earlougher, R., "Advances in Well Test Analysis," Monograph Series, SPE, Dallas (1977).

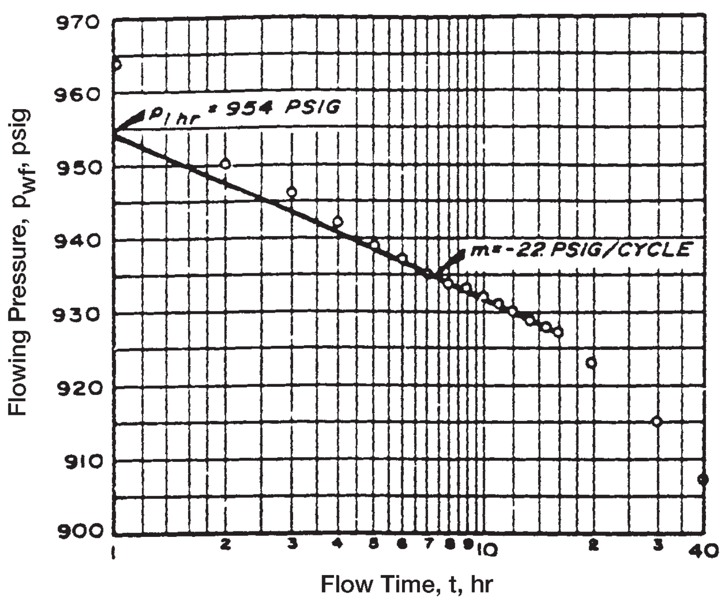



**Figure 6-34.** Earlougher's semilog data plot for the drawdown test. (*Permission to publish by the SPE, copyright SPE, 1977.*)

## Solution

*Step 1.* From Figure 6-34, calculate $p_{1\,hr}$:

$$p_{1\,hr} = 954 \text{ psi}$$

*Step 2.* Determine the slope of the transient flow line:

$$m = -22 \text{ psi/cycle}$$

*Step 3.* Calculate the permeability by applying Equation 6-173:

$$k = \frac{-(162.6)(348)(1.14)(3.93)}{(-22)(130)} = 89 \text{ md}$$

*Step 4.* Solve for the skin factor s by using Equation 6-174:

$$s = 1.151\left\{\left(\frac{954 - 1{,}154}{-22}\right) - \log\left[\frac{89}{(0.2)\,(3.93)\,(8.74\times 10^{-6})(0.25)^2}\right] + 3.2275\right\} = 4.6$$

Basically, well test analysis deals with the interpretation of the wellbore pressure response to a given change in the flow rate (from zero to a constant value for a drawdown test, or from a constant rate to zero for a buildup test). Unfortunately, the producing rate is controlled at the surface, not at the sand face. Because of the wellbore volume, a constant surface flow rate does not ensure that the entire rate is being produced from the formation. This effect is due to **wellbore storage.** Consider the case of a drawdown test. When the well is first open to flow after a shut-in period, the pressure in the wellbore drops. This drop in the wellbore pressure causes the following two types of wellbore storage:

- Wellbore storage effect caused by **fluid expansion**
- Wellbore storage effect caused by **changing fluid level** in the casing-tubing annulus.

As the bottom hole pressure drops, the wellbore fluid expands and, thus, the initial surface flow rate is not from the formation, but essentially from the fluid that had been stored in the wellbore. This is defined as the wellbore storage due to fluid expansion.

The second type of wellbore storage is due to a changing of the annulus fluid level (falling level during a drawdown test and rising fluid level during a pressure buildup test). When the well is open to flow during a drawdown test, the reduction in pressure causes the fluid level in the annulus to fall. This annulus fluid production joins that from the formation and contributes to the total flow from the well. The falling fluid level is generally able to contribute more fluid than that by expansion.

The above discussion suggests that part of the flow will be contributed by the wellbore instead of the reservoir, i.e.,

$$q = q_f + q_{wb}$$

where $q$ = surface flow rate, bbl/day
$q_f$ = formation flow rate, bbl/day

$q_{wb}$ = flow rate contributed by the wellbore, bbl/day

As production time increases, the wellbore contribution decreases, and the formation rate increases until it eventually equals the surface flow rate. During this period when the formation rate is changed, the measured drawdown pressures will not produce the ideal semilog straight-line behavior that is expected during transient flow. This indicates that the pressure data collected during the duration of the wellbore storage effect cannot be analyzed by using conventional methods.

Each of the above two effects can be quantified in terms of the wellbore storage factor C, which is defined as:

$$C = \frac{\Delta V_{wb}}{\Delta p}$$

where C = wellbore storage volume, bbl/psi
$\Delta V_{wb}$ = change in the volume of fluid in the wellbore, bbl

The above relationship can be applied to mathematically represent the individual effect of wellbore fluid expansion and falling (or rising) fluid level, to give:

- **Wellbore Storage Effect Due to Fluid Expansion**

$$C = V_{wb}\, c_{wb}$$

where $V_{wb}$ = total wellbore fluid volume, bbl
$c_{wb}$ = average compressibility of fluid in the wellbore, $psi^{-1}$

- **Wellbore Storage Effect Due to Changing Fluid Level**

If $A_a$ is the cross-sectional area of the annulus, and ρ is the average fluid density in the wellbore, the wellbore storage coefficient is given by:

$$C = \frac{144\ A_a}{5.615\,\rho}$$

with:

$$A_a = \frac{\pi\left[(ID_C)^2 - (OD_T)^2\right]}{4\ (144)}$$

where $A_a$ = annulus cross-sectional area, $ft^2$

$OD_T$ = outside diameter of the production tubing, in.
$ID_C$ = inside diameter of the casing, in.
$\rho$ = wellbore fluid density, lb/ft$^3$

This effect is essentially small if a packer is placed near the producing zone. The total storage effect is the sum of both effects. It should be noted during oil well testing that the fluid expansion is generally insignificant due to the small compressibility of liquids. For gas wells, the primary storage effect is due to gas expansion.

To determine the duration of the wellbore storage effect, it is convenient to express the wellbore storage factor in a dimensionless form as:

$$C_D = \frac{5.615\,C}{2\,\pi\,h\phi\,c_t\,r_w^2} = \frac{0.894\,C}{\phi\,h\,c_t\,r_w^2}$$

where $C_D$ = dimensionless wellbore storage factor
$C$ = wellbore storage factor, bbl/psi
$c_t$ = total compressibility coefficient, psi$^{-1}$
$r_w$ = wellbore radius, ft
$h$ = thickness, ft

Horne (1995) and Earlougher (1977), among other authors, have indicated that the wellbore pressure is directly proportional to the time during the wellbore storage-dominated period of the test and is expressed by:

$$p_D = t_D/C_D$$

where $p_D$ = dimensionless pressure during wellbore storage domination time
$t_D$ = dimensionless time

Taking the logarithm of both sides of the above relationship, gives:

$$\log(p_D) = \log(t_D) - \log(C_D)$$

The above expression has a characteristic that is diagnostic of wellbore storage effects. It indicates that a plot of $p_D$ versus $t_D$ on a log-log scale will yield as straight line of a unit slope during wellbore storage domination. Since $p_D$ is proportional to $\Delta p$ and $t_D$ is proportional to time, it is convenient to log $(p_i - p_{wf})$ versus log (t) and observe where the plot has a slope of one cycle in pressure per cycle in time.

The log-log plot is a valuable aid for recognizing wellbore storage effects in transient tests (e.g., drawdown or buildup tests) when early-time pressure recorded data are available. It is recommended that this plot be made a part of transient test analysis. As wellbore storage effects become less severe, the formation begins to influence the bottom-hole pressure more and more, and the data points on the log-log plot fall below the unit-slope straight line and signify the end of the wellbore storage effect. At this point, wellbore storage is no longer important and standard semilog data-plotting analysis techniques apply. As a rule of thumb, that time usually occurs about 1 to 1½ cycles in time after the log-log data plot starts deviating significantly from the unit slop. This time may be estimated from:

$$t_D > (60 + 3.5s)\, C_D$$

or approximately:

$$t > \frac{(200{,}000 + 12{,}000\, s)C}{(kh / \mu)}$$

where t = total time that marks the end of wellbore storage effect and the beginning of the semilog straight line, hr
k = permeability, md
s = skin factor
m = viscosity, cp
C = wellbore storage coefficient, bbl/psi

## Example 6-25

The following data are given for an oil well that is scheduled for a drawdown test:

- Volume of fluid in the wellbore = 180 bbls
- Tubing outside diameter = 2 inches
- Production casing inside diameter = 7.675 inches
- Average oil density in the wellbore = 45 lb/ft$^3$
- $h = 20$ ft, $\phi = 15\%$, $r_w = 0.25$ ft
  $\mu_o = 2$ cp, $k = 30$ md, $s = 0$
  $c_t = 20 \times 10^{-6}$ psi$^{-1}$, $c_o = 10 \times 10^{-6}$ psi$^{-1}$

If this well is placed under a constant production rate, how long will it take for wellbore storage effects to end?

## Solution

*Step 1.* Calculate the cross-sectional area of the annulus $A_a$:

$$A_a = \frac{\pi\,[(7.675)^2 - (2)^2]}{(4)\,(144)} = 0.2995\ \text{ft}^2$$

*Step 2.* Calculate the wellbore storage factor caused by fluid expansion:

$$C = V_{wb}\, c_{wb}$$
$$C = (180)\,(10 \times 10^{-6}) = 0.0018\ \text{bbl/psi}$$

*Step 3.* Determine the wellbore storage factor caused by the falling fluid level:

$$C = \frac{144\ A_a}{5.615\,\rho}$$

$$C = \frac{144\,(0.2995)}{(5.615)\,(45)} = 0.1707\ \text{bbl/psi}$$

*Step 4.* Calculate the total wellbore storage coefficient:

$$C = 0.0018 + 0.1707 = 0.1725\ \text{bbl/psi}$$

The above calculations show that the effect of fluid expansion can generally be neglected in crude oil systems.

*Step 5.* Determine the time required for wellbore storage influence to end from:

$$t = \frac{(200{,}000 + 12{,}000\ s)\ C\,\mu}{kh}$$

$$t = \frac{(200{,}000 + 0)\,(0.1725)\,(2)}{(30)\,(20)} = 115\ \text{hr}$$

The straight-line relationship as expressed by Equation 6-171 is only valid during the infinite-acting behavior of the well. Obviously, reservoirs are not infinite in extent, thus the infinite-acting radial flow period cannot last indefinitely. Eventually the effects of the reservoir boundaries will be felt at the well being tested. The time at which the boundary effect is felt is dependent on the following factors:

- Permeability k
- Total compressibility $c_t$
- Porosity $\phi$
- Viscosity $\mu$
- Distance to the boundary
- Shape of the drainage area

Earlougher (1977) suggests the following mathematical expression for estimating the duration of the infinite-acting period.

$$t_{eia} = \left[\frac{\phi \mu c_t A}{0.000264\, k}\right](t_{DA})_{eia}$$

where $t_{eia}$ = time to the end of infinite-acting period, hr
$A$ = well drainage area, $ft^2$
$c_t$ = total compressibility, $psi^{-1}$
$(t_{DA})_{eia}$ = dimensionless time to the end of the infinite-acting period

Earlougher's expression can be used to predict the end of transient flow in a drainage system of any geometry by obtaining the value of $(t_{DA})_{eia}$ from Table 6-3 as listed under "**Use Infinite System Solution with Less Than 1% Error for $t_D$** <." For example, for a well centered in a circular reservoir, $(t_{DA})_{eia} = 0.1$, and accordingly:

$$t_{eia} = \frac{380 \phi \mu c_t A}{k}$$

Hence, the specific steps involved in a drawdown test analysis are:

1. Plot $(p_i - p_{wf})$ versus t on a log-log scale.
2. Determine the time at which the unit slope line ends.
3. Determine the corresponding time at 1½ log cycle, ahead of the observed time in Step 2. This is the time that marks the end of the wellbore storage effect and the start of the semilog straight line.
4. Estimate the wellbore storage coefficient from:

$$C = \frac{qt}{24\Delta p}$$

where t and Δp are values read from a point on the log-log unit-slope straight line and q is the flow rate in bbl/day.

5. Plot $p_{wf}$ versus t on a semilog scale.
6. Determine the start of the straight-line portion as suggested in Step 3 and draw the best line through the points.
7. Calculate the slope of the straight line and determine the permeability k and skin factor s by applying Equations 6-173 and 6-174, respectively.
8. Estimate the time to the end of the infinite-acting (transient flow) period, i.e., $t_{eia}$, which marks the beginning of the pseudosteady-state flow.
9. Plot all the recorded pressure data after $t_{eia}$ as a function of time on a regular Cartesian scale. These data should form a straight-line relationship.
10. Determine the **slope** of the pseudosteady-state line, i.e., dp/dt (commonly referred to as m′) and use Equation 6-128 to solve for the drainage area "A,"

$$A = \frac{-0.23396\, Q\, B}{c_t\, h\, \phi\, (dp/dt)} = \frac{-0.23396\, Q\, B}{c_t\, h\, \phi\, m'}$$

where m′ = slope of the semisteady-state Cartesian straight line
Q = fluid flow rate, STB/day
B = formation volume factor, bbl/STB

11. Calculate the shape factor $C_A$ from an expression that has been developed by Earlougher (1977). Earlougher has shown that the reservoir shape factor can be estimated from the following relationship:

$$C_A = 5.456 \left(\frac{m}{m'}\right) \exp\left[\frac{2.303\,(p_{1\,hr} - p_{int})}{m}\right]$$

where m = slope of transient semilog straight line, psi/log cycle
m′ = slope of the semisteady-state Cartesian straight line
$p_{1hr}$ = pressure at t = 1 hr from semilog straight line, psi
$p_{int}$ = pressure at t = 0 from semisteady-state Cartesian straight line, psi

12. Use Table 6-4 to determine the drainage configuration of the tested well that has a value of the shape factor $C_A$ closest to that of the calculated one, i.e., Step 11.

## Pressure Buildup Test

The use of pressure buildup data has provided the reservoir engineer with one more useful tool in the determination of reservoir behavior. Pressure buildup analysis describes the build up in wellbore pressure with time after a well has been shut-in. One of the principal objectives of this analysis is to determine the static reservoir pressure without waiting weeks or months for the pressure in the entire reservoir to stabilize. Because the buildup in wellbore pressure will generally follow some definite trend, it has been possible to extend the pressure buildup analysis to determine:

- Effective reservoir permeability
- Extent of permeability damage around the wellbore
- Presence of faults and to some degree the distance to the faults
- Any interference between producing wells
- Limits of the reservoir where there is not a strong water drive or where the aquifer is no larger than the hydrocarbon reservoir

Certainly all of this information will probably not be available from any given analysis, and the degree of usefulness of any of this information will depend on the experience in the area and the amount of other information available for correlation purposes.

The general formulas used in analyzing pressure buildup data come from a solution of the diffusivity equation. In pressure buildup and drawdown analyses, the following assumptions, with regard to the reservoir, fluid, and flow behavior, are usually made:

**Reservoir:**

- Homogeneous
- Isotropic
- Horizontal of uniform thickness

**Fluid:**

- Single phase
- Slightly compressible
- Constant $\mu_o$ and $B_o$

**Flow:**

- Laminar flow
- No gravity effects

Pressure buildup testing requires shutting in a producing well. The most common and the simplest analysis techniques require that the well produce at a constant rate, either from startup or long enough to establish a stabilized pressure distribution, before shut-in. Figure 6-35 schematically shows rate and pressure behavior for an ideal pressure buildup test. In that figure, $t_p$ is the production time and $\Delta t$ is running shut-in time. The pressure is measured immediately before shut-in and is recorded as a function of time during the shut-in period. The resulting pressure buildup curve is analyzed for reservoir properties and wellbore condition.

Stabilizing the well at a constant rate before testing is an important part of a pressure buildup test. If stabilization is overlooked or is impossible, standard data analysis techniques may provide erroneous information about the formation.

A pressure buildup test is described mathematically by using the principle of superposition. Before the shut-in, the well is allowed to flow at a constant flow rate of $Q_o$ STB/day for $t_p$ days. At the time corresponding to the point of shut-in, i.e., $t_p$, a second well, superimposed over the location of the first well, is opened to flow at a constant rate equal to $-Q_o$ STB/day for $\Delta t$ days. The first well is allowed to continue to flow at $+Q_o$ STB/day. When the effects of the two wells are added, the result is that a well has been allowed to flow at rate Q for time $t_p$ and then shut-in for time $\Delta t$. This simulates the actual test procedure. The time corresponding to the point of shut-in, $t_p$, can be estimated from the following equation:

$$t_P = \frac{24\,N_P}{Q_o} \tag{6-175}$$

where $N_p$ = well cumulative oil produced before shut-in, STB
$Q_o$ = stabilized well flow rate before shut-in, STB/day
$t_p$ = total production time, hr

Applying the superposition principle to a shut-in well, the total pressure change, i.e., $(p_i - p_{ws})$, which occurs at the wellbore during the shut-in time $\Delta t$, is essentially the sum of the pressure change caused by the constant flow rate Q and that of $-Q$, or:

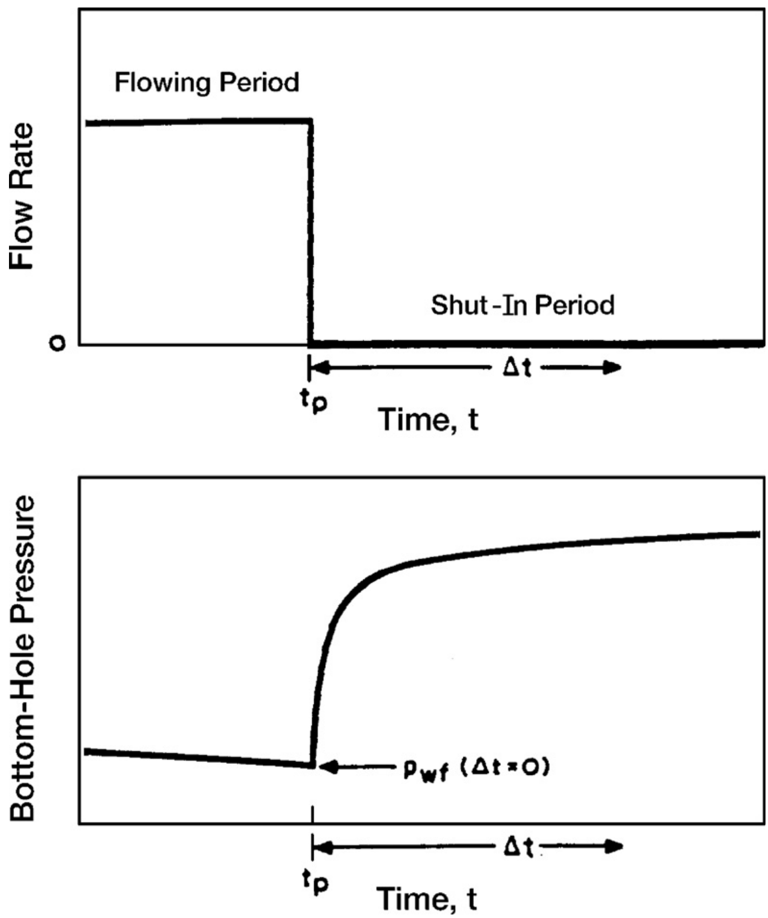



**Figure 6-35.** Idealized pressure buildup test.

$$p_i - p_{ws} = (p_i - p_{wf})Q_o - 0 + (p_i - p_{wf})0 - Q_o$$

Substituting Equation 6-146 for each of the terms on the right-hand side of the above relationship gives:

$$p_{ws} = p_i - \frac{162.6(Q_o - 0)\mu B_o}{kh}\left[\log\frac{k(t_p + \Delta t)}{\phi \mu c_t r_w^2} - 3.23 + 0.875\,s\right]$$

$$+ \frac{162.6(0 - Q_o)\mu B_o}{kh}\left[\log\frac{k(\Delta t)}{\phi \mu c_t r_w^2} - 3.23 + 0.875\,s\right] \tag{6-175}$$

Expanding this equation and canceling terms,

$$p_{ws} = p_i - \frac{162.6 Q_o \mu B}{kh}\left[\log\frac{(t_p + \Delta t)}{\Delta t}\right] \tag{6-176}$$

where $p_i$ = initial reservoir pressure, psi
$p_{ws}$ = sand-face pressure during pressure buildup, psi
$t_p$ = flowing time before shut-in, hr
$\Delta t$ = shut-in time, hr

The pressure buildup equation, i.e., Equation 6-176, was introduced by Horner (1951) and is commonly referred to as the Horner equation.

Equation 6-177 suggests that a plot of $p_{ws}$ versus $(t_p + \Delta t)/\Delta t$ would produce a straight-line relationship with intercept $p_i$ and slope of −m, where:

$$m = \frac{162.6 Q_o B_o \mu_o}{kh}$$

or

$$k = \frac{162.6 Q_o B_o \mu_o}{mh} \tag{6-177}$$

This plot, commonly referred to as the Horner plot, is illustrated in Figure 6-36. Note that on the Horner plot, the scale of time ratio increases from left to right. Because of the form of the ratio, however, the shut-in time $\Delta t$ increases from right to left. It is observed from Equation 6-177 that $p_{ws} = p_i$ when the time ratio is unity. Graphically this means that the initial reservoir pressure, $p_i$, can be obtained by extrapolating the Horner plot straight line to $(t_p + \Delta t)/\Delta t = 1$.

Earlougher (1977) points out that a result of using the superposition principle is that skin factor, s, does not appear in the general pressure buildup equation, Equation 6-176. As a result, skin factor does not appear in the simplified equation for the Horner plot, Equation 6-177. That means the Horner-plot slope is not affected by the skin factor; however, the skin factor still does affect the shape of the pressure buildup data. In fact, an early-time deviation from the straight line can be caused by skin factor as well as by wellbore storage, as indicated in Figure 6-36. The deviation can be significant for the large negative skins that occur in hydraulically fractured wells. In any case, the skin factor does affect

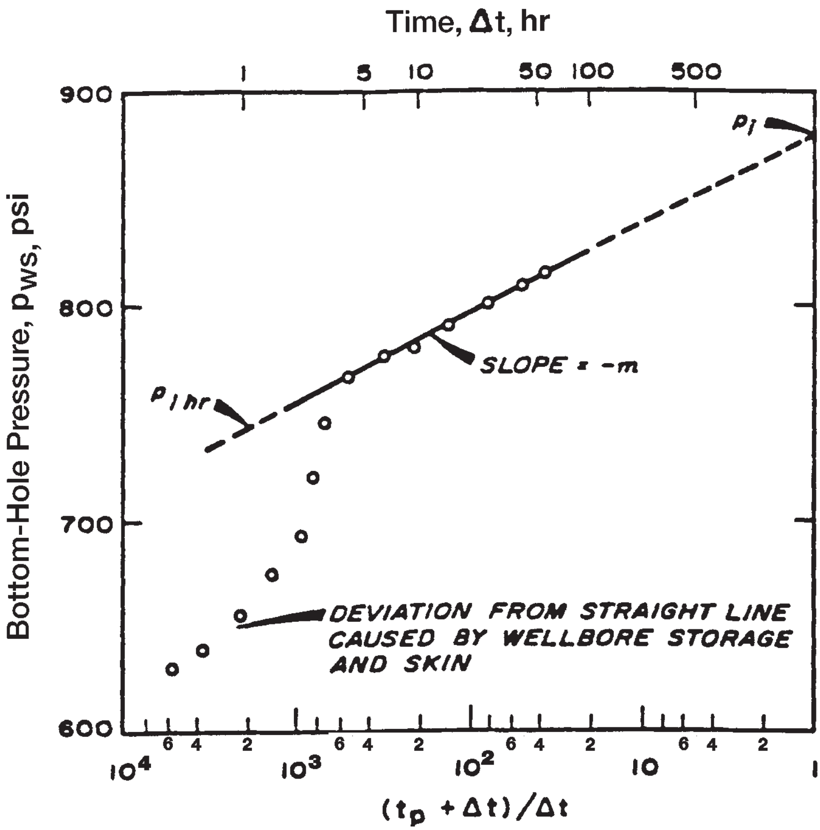



**Figure 6-36.** Horner plot. (*After Earlougher, R.* "Advances in Well Test Analysis.") (*Permission to publish by the SPE, copyright SPE, 1977.*)

flowing pressure before shut-in, so skin may be estimated from the buildup test data plus the flowing pressure immediately before the buildup test:

$$s = 1.151\left[\frac{p_{1hr} - p_{wf}(\Delta t = 0)}{m} - \log\frac{k}{\phi \mu c_t r_w^2} + 3.23\right] \qquad (6\text{-}178)$$

where $p_{wf}\,(\Delta t = 0)$ = observed flowing bottom-hole pressure immediately before shut-in

m = slope of the Horner plot

k = permeability, md

$$\Delta p_{skin} = 0.87 \text{ m s} \tag{6-179}$$

The value of $p_{1\,hr}$ must be taken from the Horner straight line. Frequently, pressure data do not fall on the straight line at 1 hour because of wellbore storage effects or large negative skin factors. In that case, the semilog line must be extrapolated to 1 hour and the corresponding pressure is read.

It should be pointed out that when a well is shut in for a pressure buildup test, the well is usually closed at the surface rather than the sand face. Even though the well is shut-in, the reservoir fluid continues to flow and accumulates in the wellbore until the well fills sufficiently to transmit the effect of shut-in to the formation. This "after-flow" behavior is caused by the wellbore storage, and it has a significant influence on pressure buildup data. During the period of wellbore storage effects, the pressure data points fall below the semilog straight line. The duration of those effects may be estimated by making the log-log data plot described previously. For pressure buildup testing, plot log $[p_{ws} - p_{wf}]$ versus log $(\Delta t)$. The bottom-hole flow pressure $p_{wf}$ is observed flowing pressure immediately before shut-in. When wellbore storage dominates, that plot will have a unit-slope straight line; as the semilog straight line is approached, the log-log plot bends over to a gently curving line with a low slope.

In all pressure buildup test analyses, the log-log data plot should be made before the straight line is chosen on the semilog data plot. This log-log plot is essential to avoid drawing a semilog straight line through the wellbore storage-dominated data. The beginning of the semilog line can be estimated by observing when the data points on the log-log plot reach the slowly curving low-slope line and adding 1 to 1.5 cycles in time after the end of the unit-slope straight line. Alternatively, the time to the beginning of the semilog straight line can be estimated from:

$$\Delta t > \frac{170{,}000\ C e^{0.14 s}}{(kh/\mu)}$$

where $\Delta t$ = shut-in time, hr
C = calculated wellbore storage coefficient, bbl/psi
k = permeability, md
s = skin factor
h = thickness, ft

## Example 6-26[3]

Table 6-5 shows pressure buildup data from an oil well with an estimated drainage radius of 2,640 ft.

**Table 6-5**
**Earlougher's Pressure Buildup Data**
**(*Permission to publish by the SPE, copyright SPE, 1977*)**

| $\Delta t$ (hours) | $t_p + \Delta t$ (hours) | $\frac{(t_p + \Delta t)}{\Delta t}$ | $p_{ws}$ (psig) |
|---|---|---|---|
| 0.0 | — | — | 2761 |
| 0.10 | 310.10 | 3101 | 3057 |
| 0.21 | 310.21 | 1477 | 3153 |
| 0.31 | 310.31 | 1001 | 3234 |
| 0.52 | 310.52 | 597 | 3249 |
| 0.63 | 310.63 | 493 | 3256 |
| 0.73 | 310.73 | 426 | 3260 |
| 0.84 | 310.84 | 370 | 3263 |
| 0.94 | 310.94 | 331 | 3266 |
| 1.05 | 311.05 | 296 | 3267 |
| 1.15 | 311.15 | 271 | 3268 |
| 1.36 | 311.36 | 229 | 3271 |
| 1.68 | 311.68 | 186 | 3274 |
| 1.99 | 311.99 | 157 | 3276 |
| 2.51 | 312.51 | 125 | 3280 |
| 3.04 | 313.04 | 103 | 3283 |
| 3.46 | 313.46 | 90.6 | 3286 |
| 4.08 | 314.08 | 77.0 | 3289 |
| 5.03 | 315.03 | 62.6 | 3293 |
| 5.97 | 315.97 | 52.9 | 3297 |
| 6.07 | 316.07 | 52.1 | 3297 |
| 7.01 | 317.01 | 45.2 | 3300 |
| 8.06 | 318.06 | 39.5 | 3303 |
| 9.00 | 319.00 | 35.4 | 3305 |
| 10.05 | 320.05 | 31.8 | 3306 |
| 13.09 | 323.09 | 24.7 | 3310 |
| 16.02 | 326.02 | 20.4 | 3313 |
| 20.00 | 330.00 | 16.5 | 3317 |
| 26.07 | 336.07 | 12.9 | 3320 |
| 31.03 | 341.03 | 11.0 | 3322 |
| 34.98 | 344.98 | 9.9 | 3323 |
| 37.54 | 347.54 | 9.3 | 3323 |

[3]This example problem and solution procedure are given by Earlougher, R., "Advanced Well Test Analysis," Monograph Series, SPE, Dallas (1977).

Before shut-in, the well had produced at a stabilized rate of 4,900 STB/day for 310 hours. Known reservoir data are:

$$
\begin{aligned}
r_e &= 2640 \text{ ft} \\
\text{depth} &= 10476 \text{ ft} \\
r_w &= 0.354 \text{ ft} \\
c_t &= 22.6 \times 10^{-6} \text{ psi}^{-1} \\
Q_o &= 4900 \text{ STB/D} \\
h &= 482 \text{ ft} \\
p_{wf}(\Delta t = 0) &= 2761 \text{ psig} \\
\mu_o &= 0.20 \text{ cp} \\
\phi &= 0.09 \\
B_o &= 1.55 \text{ bbl/STB} \\
\text{casing ID} &= 0.523 \text{ ft} \\
t_p &= 310 \text{ hours}
\end{aligned}
$$

Calculate

- Average permeability k
- Skin factor
- Pressure drop due to skin

## Solution

*Step 1.* Plot $p_{ws}$ versus $(t_p + \Delta t)/\Delta t$ on a semilog scale as shown in Figure 6-37.

*Step 2.* Identify the correct straight-line portion of the curve and determine the slope m to give:

$$m = 40 \text{ psi/cycle}$$

*Step 3.* Calculate the average permeability by using Equation 6-178 to give:

$$k = \frac{(162.6)\,(4,900)\,(1.55)\,(0.22)}{(40)\,(482)} = 12.8 \text{ md}$$

*Step 4.* Determine $p_{wf}$ after 1 hour from the straight-line portion of the curve to give:

$$p_{1\text{ hr}} = 3266 \text{ psi}$$

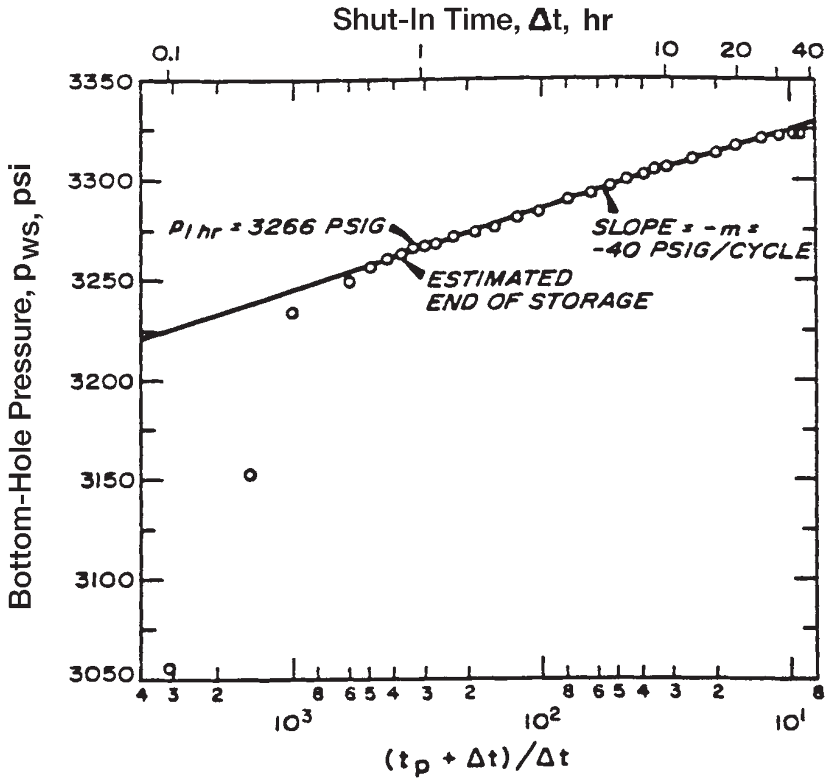



**Figure 6-37.** Earlougher's semilog data plot for the buildup test. (*Permission to publish by the SPE, copyright SPE, 1977.*)

*Step 5.* Calculate the skin factor by applying Equation 6-179.

$$s = 1.1513\left[\frac{3,266-2,761}{40} - \log\left(\frac{(12.8)\,(12)^2}{(0.09)\,(0.20)\,(22.6\times10^{-6})(4.25)^2}\right)+3.23\right] = 8.6$$

*Step 6.* Calculate the pressure drop due to skin from:

$\Delta p_{skin} = 0.87ms = 0.87\ (40)\ (8.6) = 299$ psia

## PROBLEMS

1. An incompressible fluid flows in a linear porous media with the following properties.

| | | |
|---|---|---|
| L = 2500 ft | h = 30 ft | width = 500 ft |
| k = 50 md | $\phi$ = 17% | $\mu$ = 2 cp |
| inlet pressure = 2100 psi | Q = 4 bbl/day | $\rho$ = 45 lb/ft$^3$ |

   Calculate and plot the pressure profile throughout the linear system.
2. Assume the reservoir linear system as described in problem 1 is tilted with a dip angle of 7°. Calculate the fluid potential through the linear system.
3. A 0.7 specific gravity gas is flowing in a linear reservoir system at 150°F. The upstream and downstream pressures are 2,000 and 1,800 psi, respectively. The system has the following properties:

| | | |
|---|---|---|
| L = 2000 ft | W = 300 ft | h = 15 ft |
| k = 40 md | $\phi$ = 15% | |

   Calculate the gas flow rate.
4. An oil well is producing a crude oil system at 1,000 STB/day and 2,000 psi of bottom-hole flowing pressure. The pay zone and the producing well have the following characteristics:

| | | |
|---|---|---|
| h = 35 ft | $r_w$ = 0.25 ft | drainage area = 40 acres |
| API = 45° | $\gamma_g$ = 0.72 | $R_s$ = 700 scf/STB |
| k = 80 md | T = 100°F | |

   Assuming steady-state flowing conditions, calculate and plot the pressure profile around the wellbore.
5. Assuming steady-state flow and incompressible fluid, calculate the oil flow rate under the following conditions:

| | | |
|---|---|---|
| $p_e$ = 2500 psi | $p_{wf}$ = 2000 psi | $r_e$ = 745 ft |
| $r_w$ = 0.3 ft | $\mu_o$ = 2 cp | $B_o$ = 1.4 bbl/STB |
| h = 30 ft | k = 60 md | |

6. A gas well is flowing under a bottom-hole flowing pressure of 900 psi. The current reservoir pressure is 1,300 psi. The following additional data are available:

| | | |
|---|---|---|
| $T = 140°F$ | $\gamma_g = 0.65$ | $r_w = 0.3$ ft |
| $k = 60$ md | $h = 40$ ft | $r_e = 1000$ ft |

Calculate the gas flow rate by using a:
- Real gas pseudo pressure approach
- Pressure-squared method

7. An oil well is producing a stabilized flow rate of 500 STB/day under a transient flow condition. Given:

| | | |
|---|---|---|
| $B_o = 1.1$ bbl/STB | $\mu_o = 2$ cp | $c_t = 15 \times 10^{-6}$ psi$^{-1}$ |
| $k_o = 50$ md | $h = 20$ ft | $\phi = 20\%$ |
| $r_w = 0.3$ ft | $p_i = 3500$ psi | |

Calculate and plot the pressure profile after 1, 5, 10, 15, and 20 hours.

8. An oil well is producing at a constant flow rate of 800 STB/day under a transient flow condition. The following data are available:

| | | |
|---|---|---|
| $B_o = 1.2$ bbl/STB | $\mu_o = 3$ cp | $c_t = 15 \times 10^{-6}$ psi$^{-1}$ |
| $k_o = 100$ md | $h = 25$ ft | $\phi = 15\%$ |
| $r_w = 0.5$ | $p_i = 4000$ psi | $r_e = 1000$ ft |

Using the $E_i$-function approach and the $p_D$-method, calculate the bottom-hole flowing pressure after 1, 2, 3, 5, and 10 hr. Plot the results on a semi log scale and Cartesian scale.

9. A well is flowing under a drawdown pressure of 350 psi and produces at a constant flow rate of 300 STB/day. The net thickness is 25 ft. Given:

| | | | |
|---|---|---|---|
| $r_e = 660$ ft | $r_w = 0.25$ ft | $\mu_o = 1.2$ cp | $B_o = 1.25$ bbl/STB |

Calculate:
- Average permeability
- Capacity of the formation

10. An oil well is producing from the center of 40-acre-square drilling pattern. Given:

| | | |
|---|---|---|
| $\phi = 20\%$ | $h = 15$ ft | $k = 60$ md |
| $\mu_o = 1.5$ cp | $B_o = 1.4$ bbl/STB | $r_w = 0.25$ ft |
| $p_r = 2000$ psi | $p_{wf} = 1500$ psi | |

Calculate the oil flow rate.

11. A shut-in well is located at a distance of 700 ft from one well and 1100 ft from a second well. The first well flows for 5 days at 180 STB/day, at which time the second well begins to flow at 280 STB/day. Calculate the pressure drop in the shut-in well when the second well has been flowing for 7 days. The following additional data are given:

| | | | |
|---|---|---|---|
| $p_i = 3000$ psi | $B_o = 1.3$ bbl/STB | $\mu_o = 1.2$ cp | $h = 60$ ft |
| $c_t = 15 \times 10^{-6}$ psi$^{-1}$ | $\phi = 15\%$ | $k = 45$ md | |

12. A well is opened to flow at 150 STB/day for 24 hours. The flow rate is then increased to 360 STB/day and lasted for another 24 hours. The well flow rate is then reduced to 310 STB/day for 16 hours. Calculate the pressure drop in a shut-in well 700 ft away from the well given:

| | | |
|---|---|---|
| $\phi = 15\%$ | $h = 20$ ft | $k = 100$ md |
| $\mu_o = 2$ cp | $B_o = 1.2$ bbl/STB | $r_w = 0.25$ ft |
| $p_i = 3000$ psi | $c_t = 12 \times 10^{-6}$ psi$^{-1}$ | |

13. A well is flowing under unsteady-state flowing conditions for 5 days at 300 STB/day. The well is located at 350 ft and 420 ft distance from two sealing faults. Given:

| | | |
|---|---|---|
| $\phi = 17\%$ | $c_t = 16 \times 10^{-6}$ psi$^{-1}$ | $k = 80$ md |
| $p_i = 3000$ psi | $B_o = 1.3$ bbl/STB | $\mu_o = 1.1$ cp |
| $r_w = 0.25$ ft | $h = 25$ ft | |

Calculate the pressure in the well after 5 days.

14. A drawdown test was conducted on a new well with results as given below:

| t, hr | $p_{wf}$, psi |
|---|---|
| 1.50 | 2978 |
| 3.75 | 2949 |
| 7.50 | 2927 |
| 15.00 | 2904 |
| 37.50 | 2876 |
| 56.25 | 2863 |
| 75.00 | 2848 |
| 112.50 | 2810 |
| 150.00 | 2790 |
| 225.00 | 2763 |

Given:

$p_i = 3400$ psi  $h = 25$ ft  $Q = 300$ STB/day
$c_t = 18 \times 10^{-6}$ psi$^{-1}$  $\mu_o = 1.8$ cp  $B_o = 1.1$ bbl/STB
$r_w = 0.25$ ft  $\phi = 12\%$

Assuming no wellbore storage, calculate:

- Average permeability
- Skin factor

15. A drawdown test was conducted on a discovery well. The well was flowed at a constant flow rate of 175 STB/day. The fluid and reservoir data are given below:

$S_{wi} = 25\%$  $\phi = 15\%$  $h = 30$ ft  $ct = 18 \times 10^{-6}$ psi$^{-1}$
$r_w = 0.25$ ft  $p_i = 4680$ psi  $\mu_o = 1.5$ cp  $B_o = 1.25$ bbl/STB

The drawdown test data are given below:

| t, hr | $p_{wf}$, psi |
|---|---|
| 0.6 | 4388 |
| 1.2 | 4367 |
| 1.8 | 4355 |
| 2.4 | 4344 |
| 3.6 | 4334 |
| 6.0 | 4318 |
| 8.4 | 4309 |
| 12.0 | 4300 |
| 24.0 | 4278 |
| 36.0 | 4261 |
| 48.0 | 4258 |
| 60.0 | 4253 |
| 72.0 | 4249 |
| 84.0 | 4244 |
| 96.0 | 4240 |
| 108.0 | 4235 |
| 120.0 | 4230 |
| 144.0 | 4222 |
| 180.0 | 4206 |

Calculate:

- Drainage radius
- Skin factor
- Oil flow rate at a bottom-hole flowing pressure of 4,300 psi, assuming a semisteady-state flowing condition.

16. A pressure build up test was conducted on a well that had been producing at 146 STB/day for 53 hours. The reservoir and fluid data are given below.

| | | |
|---|---|---|
| $B_o = 1.29$ bbl/STB | $\mu_o = 0.85$ cp | $c_t = 12 \times 10^{-6}$ psi$^{-1}$ |
| $\phi = 10\%$ | $p_{wf} = 1426.9$ psig | A = 20 acres |

The build up data are as follows:

| Time, hr | $p_{ws}$, psig |
|---|---|
| 0.167 | 1451.5 |
| 0.333 | 1476.0 |
| 0.500 | 1498.6 |
| 0.667 | 1520.1 |
| 0.833 | 1541.5 |
| 1.000 | 1561.3 |
| 1.167 | 1581.9 |
| 1.333 | 1599.7 |
| 1.500 | 1617.9 |
| 1.667 | 1635.3 |
| 2.000 | 1665.7 |
| 2.333 | 1691.8 |
| 2.667 | 1715.3 |
| 3.000 | 1736.3 |
| 3.333 | 1754.7 |
| 3.667 | 1770.1 |
| 4.000 | 1783.5 |
| 4.500 | 1800.7 |
| 5.000 | 1812.8 |
| 5.500 | 1822.4 |
| 6.000 | 1830.7 |
| 6.500 | 1837.2 |
| 7.000 | 1841.1 |
| 7.500 | 1844.5 |
| 8.000 | 1846.7 |
| 8.500 | 1849.6 |
| 9.000 | 1850.4 |
| 10.000 | 1852.7 |
| 11.000 | 1853.5 |
| 12.000 | 1854.0 |
| 12.667 | 1854.0 |
| 14.620 | 1855.0 |

Calculate:

- Average reservoir pressure
- Skin factor
- Formation capacity

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