6.1 The Weibull Distribution

The bathtub curve in Figure 2.7 showed that, as well as random failures, there are distributions involving increasing or decreasing failure rate. In these variable failure rate cases there is no point in considering the failure rate since it is continually changing. Only reliability and MTBF are meaningful. In Chapter 2 we saw that:

\[ R(t) = \exp \left[ - \int_{0}^{t} \lambda(t) dt \right] \]

Only the random failures case enabled us to simplify this to \( R(t) = e^{-\lambda t} \) and to make use of the failure rate parameter. Since the relationship between failure rate and time can take many forms, and depends on the device/component in question, the above integral is of little direct use. Even if the variation of failure rate with time were known, it might well be of such a complex nature that the integration would prove far from simple.

In practice it is found that the distribution can usually be described by the following three-parameter distribution known as the Weibull distribution, named after Professor Waloddi Weibull:

\[ R(t) = \exp \left[ - \left( \frac{t - \gamma}{\eta} \right)^{\beta} \right] \]

In fact, for the majority of cases, a two-parameter model proves sufficient to describe the data. Hence:

\[ R(t) = \exp \left[ - \left( \frac{t}{\eta} \right)^{\beta} \right] \]

The constant failure rate case is therefore a special one-parameter case of the Weibull distribution (with \( \beta = 1 \)). However, as we have seen, it is only randomness that can be described by a single parameter (i.e. failure rate).

The three parameters (\( \gamma, \beta, \eta \)) do not have physical meanings in the same way as does failure rate. They are parameters that allow us to compute reliability and MTBF. In the special case of \( \gamma = 0 \) and \( \beta = 1 \) the expression reduces to the exponential case with \( \eta \) giving the MTBF. In the general case, however, \( \eta \) is not the MTBF and is known as the scale parameter. \( \beta \) is known as
the shape parameter and describes the rate of change of failure rate (increasing or decreasing). 
\( \gamma \) is known as the location parameter, in other words a displacement of the time origin. \( \gamma = 0 \) 
means that the time origin is, in fact, at \( t = 0 \).

The following explanation shows how data, described by a Weibull function, can be made to fit a straight line. It is not essential to follow the explanation and the reader may, if desired, move to the next block of text.

The Weibull expression can be reduced to a straight-line equation by taking logarithms twice:

If \( 1 - R(t) = Q(t) \) ... the unreliability (probability of failure in \( t \))
then

\[
1 - Q(t) = \exp \left[ - \left( \frac{t - \gamma}{\eta} \right)^\beta \right]
\]

so that

\[
\frac{1}{1 - Q(t)} = \exp \left( \frac{t - \gamma}{\eta} \right)^\beta
\]

Therefore

\[
\log \frac{1}{1 - Q(t)} = \left( \frac{t - \gamma}{\eta} \right)^\beta
\]

and

\[
\log \log \frac{1}{1 - Q(t)} = \beta \log(t - \gamma) - \beta \log \eta
\]

which is \( Y = mX + C \), the equation of a straight line.

If \( (t - \gamma) \) is replaced by \( t' \) then:

\[
Y = \log \log \frac{1}{1 - Q(t)} \quad \text{and} \quad X = \log t' \quad \text{and the slope } m = \beta.
\]

If \( Y = 0 \)

\[
\log \log \frac{1}{1 - Q(t)} = 0
\]

then

\[
\beta \log t' = \beta \log \eta
\]

so that

\[
t' = \eta
\]

This occurs if
Variable Failure Rates and Probability Plotting

\[
\log \log \frac{1}{1 - Q(t)} = 0 \quad \text{so that} \quad \log \frac{1}{1 - Q(t)} = 1
\]

i.e.

\[
\frac{1}{1 - Q(t)} = e \quad \text{and} \quad Q(t) = 0.63
\]

If a group of failure times can be modeled by a Weibull function, and it is initially assumed that \( \gamma = 0 \), then by plotting the times to failure against time on double logarithmic paper (failure percentage on loglog scale and time on log scale), a straight line should be obtained. The three Weibull parameters and hence the expression for reliability may then be obtained from measurements of the slope and intercept.

Figure 6.1 is loglog by log graph paper with suitable scales for cumulative percentage failure and time. Cumulative percentage failure is effectively the unreliability and is estimated by taking each failure in turn from median ranking tables of the appropriate sample size. It should be noted that the sample size, in this case, is the number of failures observed. However, a test yielding 10 failures from 25 items would require the first 10 terms of the median ranking table for sample size 25.

6.2 Using the Weibull Method

6.2.1 Curve Fitting to Interpret Failure Data

Assume that the failure rate is not constant OR, alternatively, that we want to determine whether or not it is constant.

Whereas in the case of random failures (dealt with in Chapter 5) it was only necessary to know the total time \( T \) applying to the \( k \) failures, it is now essential to know the individual times to failure of the items. Without this information it would not be possible to fit the data to a distribution.

The Weibull technique assumes, initially, that the distribution of failures, whilst not random, is at least able to be modeled by a simple two-parameter distribution. It assumes that:

\[
R(t) = \exp - (t/\eta)^\beta
\]

The technique is to carry out a curve-fitting (probability-modeling) exercise first to establish that the data will fit this assumption and second to estimate the values of the two parameters. Traditionally this was done by ‘pencil and paper’ curve-fitting methods, which are described here. Later in this chapter a software tool for performing this task is also described.
Figure 6.1: Graph paper for Weibull plot
If $\beta = 1$ then the failures are random and a constant failure rate can be assumed where failure rate $= 1/\eta$.

If $\beta > 1$ then the failure rate is increasing.

If $\beta < 1$ then the failure rate is decreasing.

In some cases, where the two-parameter distribution is inadequate to model the data, the three-parameter version can be used. In that case:

$$R(t) = \exp \left[ \frac{(t - \gamma)/\eta}{\beta} \right]$$

$\gamma$ can be estimated by successive iteration until a fit to the two-parameter distribution is obtained. This will be described in Section 6.3.

### 6.2.2 Manual Plotting

Ten devices were put on test and permitted to fail without replacement. The time at which each device failed was noted and from the test information we are required to determine:

1. if there is a Weibull distribution that fits these data;
2. if so, the values of $\gamma$, $\eta$ and $\beta$;
3. the probability of items surviving for specified lengths of time;
4. if the failure rate is increasing, decreasing or constant;
5. the MTBF.

The results are shown in Table 6.1 against the median ranks for sample size 10. The 10 points are plotted on Weibull paper as in Figure 6.2 and a straight line is obtained.

<table>
<thead>
<tr>
<th>Cumulative failures, $Q(t)$ (%) median rank</th>
<th>6.7</th>
<th>16.2</th>
<th>25.9</th>
<th>35.6</th>
<th>45.2</th>
<th>54.8</th>
<th>64.5</th>
<th>74.1</th>
<th>83.8</th>
<th>93.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time, $t$ (hours × 100)</td>
<td>1.7</td>
<td>3.5</td>
<td>5.0</td>
<td>6.4</td>
<td>8.0</td>
<td>9.6</td>
<td>11.</td>
<td>13.</td>
<td>18.</td>
<td>22.</td>
</tr>
</tbody>
</table>

The straight line tells us that the Weibull distribution is applicable and the parameters are determined as follows:

$\gamma$: It was shown in Section 6.1 that if the data yield a straight line then $\gamma = 0$.

$\beta$: The slope yields the value of $\beta$, which is obtained by taking a line parallel to the data line but through the origin of the construction in Figure 6.2. The value of $\beta$ is shown by the intersection with the arc. Here $\beta = 1.5$.

$\eta$: We have already shown that $\eta = t$ for $Q(t) = 0.63$, hence $\eta$ is obtained by taking a horizontal line from the origin of the construction across to the data line and then reading the corresponding value of $t$. 
The reliability expression is therefore:

$$R(t) = \exp \left[ -\left( \frac{t}{1110} \right)^{1.5} \right]$$

The probability of survival to $t = 1000$ hrs is therefore:

$$R(1000) = e^{-0.855} = 42.5\%$$

The test indicates wearout since $\beta$, which is known as the shape parameter, $> 1$.
- For increasing failure rate $\beta > 1$
- for decreasing failure rate $\beta < 1$
- for constant failure rate $\beta = 1$.

It now remains to evaluate the MTBF. This is, of course, the integral from zero to infinity of $R(t)$. Table 6.2 enables us to short-cut this step.

Since $\beta = 1.5$ then $\text{MTBF} / \eta = 0.903$ and $\text{MTBF} = 0.903 \times 1110 = 1002$ hrs. Since median rank tables have been used, the MTBF and reliability values calculated are at the 50% confidence level. In the example, time was recorded in hours but there is no reason why a more appropriate scale should not be used such as number of operations or cycles. The MTBF would then be quoted as mean number of cycles between failures.

For samples of other than 10 items, a set of median ranking tables is required. Since space does not permit a full set to be included the following approximation is given. For sample size $N$ the $r$th rank is obtained from Bernard’s approximation:
Care must be taken in the choice of the appropriate ranking table. $N$ is the number of items in the test and $r$ the number that failed, in other words, the number of data points. In our example $N$ was 10 not because the number of failures was 10 but because it was the sample size. As it happens, we considered the case where all 10 failed.

Had there been 20 items, of which 10 did not fail, the median ranks from Bernard’s formula would have been:

%: 3.4  8.3  13  18  23  28  33  38  43  48

Although this method allows for the ranking of the failures it does not take account of the actual hours contributed by the censored items. In the next section, the maximum likelihood technique is introduced partly for this purpose.

### 6.2.3 Using the COMPARE Computer Tool

The COMPARE software package provides a tool for probability plotting whereby Weibull parameters are found that best fit the data being analyzed.

Repair times and censored data are entered and estimates of the Weibull parameters, as well as a graphical plot, are provided. The term ‘censored data’ refers to items that have not failed but, nevertheless, whose operating time needs to be taken account of. There are four types of censoring:

1. Items that continue after the last failure (the most usual type of censored data).
2. Items removed (for some reason other than failure) before the test finishes.
3. Items that are added after the beginning of the test and whose operating hours need to be included.

### Table 6.2

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\frac{MTBF}{\eta}$</th>
<th>$\beta$</th>
<th>$\frac{MTBF}{\eta}$</th>
<th>$\beta$</th>
<th>$\frac{MTBF}{\eta}$</th>
<th>$\beta$</th>
<th>$\frac{MTBF}{\eta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>$\infty$</td>
<td>1.0</td>
<td>1.000</td>
<td>2.0</td>
<td>0.886</td>
<td>3.0</td>
<td>0.894</td>
</tr>
<tr>
<td>0.1</td>
<td>10!</td>
<td>1.1</td>
<td>0.965</td>
<td>2.1</td>
<td>0.886</td>
<td>3.1</td>
<td>0.894</td>
</tr>
<tr>
<td>0.2</td>
<td>5!</td>
<td>1.2</td>
<td>0.941</td>
<td>2.2</td>
<td>0.886</td>
<td>3.2</td>
<td>0.896</td>
</tr>
<tr>
<td>0.3</td>
<td>9.261</td>
<td>1.3</td>
<td>0.923</td>
<td>2.3</td>
<td>0.886</td>
<td>3.3</td>
<td>0.897</td>
</tr>
<tr>
<td>0.4</td>
<td>3.323</td>
<td>1.4</td>
<td>0.911</td>
<td>2.4</td>
<td>0.886</td>
<td>3.4</td>
<td>0.898</td>
</tr>
<tr>
<td>0.5</td>
<td>2.000</td>
<td>1.5</td>
<td>0.903</td>
<td>2.5</td>
<td>0.887</td>
<td>3.5</td>
<td>0.900</td>
</tr>
<tr>
<td>0.6</td>
<td>1.505</td>
<td>1.6</td>
<td>0.897</td>
<td>2.6</td>
<td>0.888</td>
<td>3.6</td>
<td>0.901</td>
</tr>
<tr>
<td>0.7</td>
<td>1.266</td>
<td>1.7</td>
<td>0.892</td>
<td>2.7</td>
<td>0.889</td>
<td>3.7</td>
<td>0.902</td>
</tr>
<tr>
<td>0.8</td>
<td>1.133</td>
<td>1.8</td>
<td>0.889</td>
<td>2.8</td>
<td>0.890</td>
<td>3.8</td>
<td>0.904</td>
</tr>
<tr>
<td>0.9</td>
<td>1.052</td>
<td>1.9</td>
<td>0.887</td>
<td>2.9</td>
<td>0.892</td>
<td>3.9</td>
<td>0.905</td>
</tr>
</tbody>
</table>

$$r = \frac{0.3}{N + 0.4}$$
4. Failed items that, having been restored to ‘as new’ condition, then clock up further operating time.

In the latter case it is important to be satisfied that the refurbishment really is ‘as new’. If so the additional hours count from the refurbishment and can be treated as if from an extra item.

In practice it may happen that there is a time to failure for a particular failure mode. The item might be repaired ‘as new’ and continue until it fails again. IMPORTANT – if the second failure is the same mode then the time to failure is counted from the refurbishment. If the second failure is a different mode then the time to failure is the whole operating time from the commencement of the test.

It MUST be remembered, however, that any computerized algorithm will allocate parameters to any data for a given distribution. It is, therefore, important to be aware of the limitations of probability plotting.

Two methods of estimating the Weibull parameters from a set of times to failure are LEAST SQUARES and MAXIMUM LIKELIHOOD.

The least squares method is used as an initial calculation and involves calculating the hypothetical line for which the sum of the squares of the distances of the horizontal distances from the data points to the line is a minimum. The Weibull parameters, BETA and ETA, are obtained from the line. For the two-parameter Weibull distribution, the least squares estimates are obtained from:

\[
\text{BETA} = \frac{(\Sigma(Y_i)^2 - \bar{Y} \Sigma Y_i)}{(\Sigma(X_iY_i - \bar{X} \Sigma Y_i)}
\]

\[
\text{ETA} = \exp \left( \frac{X - \bar{Y}}{\text{Beta}} \right)
\]

where

\[
Y = \log_e \left[ \log_e \left[ \frac{1}{1 - F(t)} \right] \right]
\]

\[
X = \log_e t
\]

\[
t = \text{time}
\]

Because this least squares method involves treating each of the squared distances with equal importance, it favors the higher values of time. Nevertheless, the least squares estimates of BETA and ETA may well be adequate if there are very few or, better still, no censored data. Nevertheless it does not take account of the censored data involving the times with no failure (the survivors). To deal with this the maximum likelihood estimate is required.
In COMPARE, the least squares estimates of BETA and ETA are used as the most reasonable estimate for commencing the iterative process of determining maximum likelihood values that give equal weight to each data point by virtue of calculating its probability of causing the estimated parameter. The algorithm generates the Weibull BETA and ETA parameters from which the data are most likely to have come by setting up a likelihood equation, differentiating with respect to BETA and ETA, and setting this equal to zero (in other words the standard calculus method of obtaining a minimum). The process is iterated for alternate BETA and ETA estimates until the values do not significantly change.

The maximum likelihood values are then taken as the best estimates of the Weibull parameters.

A large number of data collection schemes do not readily provide the times to failure of the items in question. For example, if an assembly (such as a valve) is replaced from time to time then its identity and its time to failure and replacement might be obtainable from the data. However, it might well be the diaphragm that is eventually the item of interest. Diaphragms may have been replaced during routine maintenance and the identity of each diaphragm not recorded. Subsequent Weibull analysis of the valve diaphragm would not then be possible. Careful thought has to be given when implementing a data collection scheme as to what subsequent data analysis will take place.

As in the above example of a valve and its diaphragm each of several failure modes will have its own failure distribution for which Weibull analysis may be appropriate. It is very likely, when attempting this type of modeling, that data not fitting the two-parameter distribution actually contain more than one failure mode. Separating out the individual failure modes may permit successful Weibull modeling.

### 6.2.4 Significance of the Result

The dangers of attempting to construct a Weibull plot with too few data points should be noted. A satisfactory result will not be obtained with fewer than at least six points. Tests yielding zero, one, two and even three failures do not allow a variable failure rate to be observed. In these cases constant failure rate must be assumed and the chi-square test used, which is a valid approach provided that the information extracted is applied only to the same time range as the test.

The comparison between the results obtained from least squares and maximum likelihood estimations (described above) provides an initial feel for how good a fit the data are to the inferred Weibull parameters.

If (in addition to the confidence obtained from the physical plot) the two values of shape parameter, obtained from least squares and maximum likelihood, are in good agreement then there is a further test.
This is provided by way of the Gnedenko test, which tests for constant failure rate. This is an ‘F’ test which tests the hypothesis that the failure times are at random, i.e. $\beta = 1$. The screen will state whether or not it is valid to reject the assumption that $\beta = 1$. The lower the value of the significance per cent then the more likely it is that the failure rate is significantly different from constant.

Essentially the test compares the MTTF of the failure times as grouped either side of the middle failure time and tests for a significant difference.

If the total number of failure times is $n$, and the time of the $n/2$th failure is $T$, the two estimates are:

$$\frac{\sum_{i=1}^{n/2} t_i(n/2 \times T)}{n/2} \text{ and } \frac{\sum_{i=n/2+1}^{n} (t_i - T)}{n/2}$$

That is to say we are comparing the MTTF of the ‘first half’ of the failures and the MTTF of the ‘second half’. The ratio should be one if the failure rate is constant. If it is not then the magnitude of the ratio gives an indication of significance. The ratio follows an ‘F’ distribution and the significance level can therefore be calculated. The two values of MTTF are shown on the screen. If this test were applied to the graphical plot in Section 6.2.2, we would see that, despite a fairly good straight line, the confidence that $\beta$ is not 1 is only 32%!

It should be remembered that a small number of failure times, despite a high value of $\beta$, may not show a significant departure from the ‘random’ assumption. In practice 10 or more failure times is a minimum desirable data set for Weibull analysis. Nevertheless, engineering judgement should always be used to temper statistical analysis. The latter looks only at numbers and does not take account of known component behaviors.

Note: If a poor fit is obtained from the two-parameter model, and the plot is a simple curve rather than ‘S’-shaped or disjointed, then it is possible to attempt a three-parameter model by estimating the value of $\gamma$ described in section 6.3. The usual approach is to assume that $\gamma$ takes the value of the first failure time and to proceed, as above, with the two-parameter model to find $\eta$ and $\beta$. Successive values of $\gamma$ can be attempted, by iteration, until the two-parameter model provides a better fit. It must be remembered, however, that if the reason for a poor fit with the two-parameter model is that only a few failure times are available then the use of the three-parameter model is unlikely to improve the situation.

If the plot is ‘S’-shaped, then it is possible that two failure modes are present in the data.

In the author’s experience only a limited number of components show a significantly increasing failure rate. This is often due to the phenomenon (known as Drenick’s law) whereby a mixture of three or more failure modes will show a random failure distribution irrespective of the BETAs of the individual modes.
6.2.5 Optimum Preventive Replacement

In Chapter 3 (Figure 3.1) the concept of optimum reliability/availability was introduced. Exactly the same picture applies to replacement interval for items with a wearout characteristic (i.e. \( \beta > 1 \)). We may choose to replace an item that wears out at some arbitrary point in its life. The longer we leave the replacement then the greater is the chance of incurring the penalty costs associate with an ‘unexpected’ failure. On the other hand the more frequently we replace the item the more we spend on replacement items. There will be an optimum point at which to carry out the preventive replacement and this is dealt with in Chapter 16.3.

6.3 More Complex Cases of the Weibull Distribution

Suppose that the data in our example had yielded a curve rather than a straight line. It is still possible that the Weibull distribution applies but with \( \gamma \) greater than zero. The approach is to choose an assumed value for \( \gamma \), usually the first value of \( t \) in the data, and replot the line against \( t' \), where \( t' = t - \gamma \). The first data point is now not available and the line will be constructed from one fewer point. Should the result be a straight line then the value of \( \gamma \) is as estimated and one proceeds as before to evaluate the other two parameters. Mean time between failure is calculated as before plus the value of \( \gamma \). If, on the other hand, another curve is generated then a further value of \( \gamma \) is tried until, by successive approximations, the correct value is found. This trial-and-error method of finding \( \gamma \) is not as time-consuming as it might seem. It is seldom necessary to attempt more than four approximations of \( \gamma \) before either generating a straight line or confirming that the Weibull distribution will not fit the data. One possible reason for the Weibull distribution not applying could be the presence of more than one failure mechanism in the data. Two mechanisms are unlikely to follow the same distribution and it is important to confine the analysis to one mechanism at a time.

So far, a single-sided analysis at 50% confidence has been described. It is possible to plot the 90% confidence bands by use of the 5% and 95% rank tables. First, Table 6.3 is constructed and the confidence bands plotted as follows.

<table>
<thead>
<tr>
<th>Time, ( t ) (hours ( \times 100 ))</th>
<th>1.7</th>
<th>3.5</th>
<th>5.0</th>
<th>6.4</th>
<th>8.0</th>
<th>9.6</th>
<th>11.</th>
<th>13.</th>
<th>18.</th>
<th>22.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Median rank</td>
<td>6.7</td>
<td>16.2</td>
<td>25.9</td>
<td>35.6</td>
<td>45.2</td>
<td>54.8</td>
<td>64.5</td>
<td>74.</td>
<td>83.8</td>
<td>93.3</td>
</tr>
<tr>
<td>5% rank</td>
<td>0.5</td>
<td>3.7</td>
<td>8.7</td>
<td>15.</td>
<td>22.</td>
<td>30.</td>
<td>39.</td>
<td>49.</td>
<td>61.</td>
<td>74.</td>
</tr>
<tr>
<td>95% rank</td>
<td>26.</td>
<td>39.</td>
<td>51.</td>
<td>61.</td>
<td>70.</td>
<td>78.</td>
<td>85.</td>
<td>91.</td>
<td>96.</td>
<td>99.</td>
</tr>
</tbody>
</table>

Consider the point corresponding to the failure at 500 hrs. The two points A and B are marked on the straight line corresponding to 8.7% and 51% respectively. The median rank for this point was 25.9% and vertical lines are drawn from A and B to intersect the horizontal. These
two points lie on the confidence bands. The other points are plotted in the same way and confidence bands are produced as shown in Figure 6.3. Looking at the curves, the limits of $Q(t)$ at 1000 hrs are 30% and 85%. At 90% confidence the reliability for 1000 hrs is therefore between 15% and 70%.

### 6.4 Continuous Processes

There is a very strict limitation to the use of this Weibull method, which is illustrated by the case of filament lamps. It is well known that these do not fail at random. Indeed, they have a pronounced wearout characteristic with a $\beta$ of two. However, imagine a brand new building with brand new lamps. Due to the distribution of failures, very few will fail in the first few months, perhaps only a few in the next few months and several towards the end of the year. After several years, however, the lamps in the building will all have been replaced at different times and the number failing in any month will be approximately the same. Thus, a population of items with increasing failure rate appears as a constant failure rate system. This is an example of a continuous process, and Figure 6.4 shows the failure characteristic of a single lamp and the superimposition of successive generations.

If the intervals between failure were observed, ranked and plotted in a Weibull analysis then a $\beta$ of 1 would be obtained. Weibull analysis must not therefore be used for the times between failure within a continuous process but only for a number of items whose individual times to failure are separately recorded. It is not uncommon for people to attempt the former and obtain a totally false picture of the process.
More suitable models for this case are the reliability growth models (CUSUM and Duane) described in Chapter 12. Another is to apply the Laplace test, which provides a means of indicating if the process failure rate has a trend.

If a system exhibits a number of failures after time zero at times $x_1, x_2, x_3, \ldots, x_i$, then the test statistic for the process is

$$U = \frac{(\sum x_i/n) - (x_0/2)}{x_0\sqrt{(1/12n)}}$$

$x_0$ is the time at which the test is truncated. If $U = 0$ then there is no trend and the failure rate is not changing. If $U < 0$ then the failure rate is decreasing and if $U > 0$ it is increasing.

This test could be applied to the analysis of software failures since they are an example of a continuous repair process.

**Exercises**

1. Components, as described in the example of Section 6.2, are to be used in a system. It is required that these are preventively replaced such that there is only a 5% probability of their failing beforehand. After how many hours should each item be replaced?

2. A sample of 10 items is allowed to fail and the time for each failure is as follows:

   4, 6, 8, 11, 12, 13, 15, 17, 20, 21 (thousand hours)

   Use the Weibull paper in this chapter to determine the reliability characteristic and the MTBF.